MATH 425b MIDTERM 2 SOLUTIONS
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(1) Let $v_k = \sum_{i=1}^k h_i e_i$, and $v_0 = 0$. Then $x + v_k \in E$ for all $k \le n$, and by the Mean Value Theorem applied to the function $t \mapsto f(x + v_{k-1} + th_k e_k)$, for each k, j there is a $\theta \in (0,1)$ such that

$$\begin{aligned} |f_j(x+v_k) - f_j(x+v_{k-1})| &= |f_j(x+v_{k-1}+h_k e_k) - f_j(x+v_{k-1})| \\ &= |h_k|\ |D_k f_j(x+v_{k-1}+\theta h_k e_k)| \\ &\le M|h|. \end{aligned}$$

Therefore

$$|f_j(x+h) - f_j(x)| \le \sum_{k=1}^n |f_j(x+v_k) - f_j(v_{k-1})| \le nM|h|,$$

and then

$$|f(x+h) - f(x)| \le \sum_{j=1}^m |f_j(x+h) - f_j(x)| \le Mmn|h|.$$

(2) Let $g(x) = d(x, \varphi(x))$. Since φ is continuous, so is g. Since X is compact, the infimum of g is achieved at some x_0. Let $x_1 = \varphi(x_0)$. Suppose $d(x_0, \varphi(x_0)) = \inf_{x \in X} g(x) = a > 0$. Then $a \le d(x_1, \varphi(x_1)) = d(\varphi(x_0), \varphi(\varphi(x_0))) < d(x_0, \varphi(x_0)) = a$, a contradiction. Therefore $d(x_0, \varphi(x_0) = 0$, that is, $x_0 = \varphi(x_0)$, so x_0 is a fixed point. If $y \neq x_0$ is another fixed point, then $d(y, x_0) = d(\varphi(y), \varphi(x_0)) < d(y, x_0)$, a contradiction, so x_0 must be the unique fixed point.

(3)(a) By the triangle inequality, $\|A\| - \|B\| \le \|A - B\|$ and $\|B\| - \|A\| \le \|A - B\|$, so $\big|\|A\| - \|B\|\big| \le \|A - B\|$. Therefore for $\delta = \epsilon$, we have $\|A - B\| < \delta$ implies $\big|\|A\| - \|B\|\big| < \epsilon$.

(b) By (a) and the $\mathcal{C}'$ assumption, $\|g'(x)\|$ is a continuous function of x on D. Therefore if we take a ball G containing z with the closure $\overline{G} \subset D$, we have $\overline{G}$ compact so $\|g'(x)\|$ is bounded on $\overline{G}$, hence also on G.

(c) By the Inverse Function Theorem, there exist neighborhoods U_0, V_0 of $a, f(a)$ such that f is a $\mathcal{C}'$ bijection of U_0 to V_0 and the inverse g is $\mathcal{C}'$. By (b), if we shrink V_0 then there exists M such that $\|g'(y)\| \le M$ for all $y \in V_0$. By Theorem 9.19, for $w, z \in V_0$ we have $|g(w) - g(z)| \le M|w - z|$. Taking $x, y \in U_0$ and $w = f(x), z = f(y)$, we get $|f(x) - f(y)| \ge M^{-1}|x - y|$.

(4)(a) Define $f : \mathbb{R}^3 \to \mathbb{R}^2$ by $f(x, y, z) = (-4x^2 + y^2 + z^2, yz - 2xy)$. Then the matrix of

$f'(x,y,z)$ is

$$\begin{bmatrix} -8x & 2y & 2z \\ -2y & z-2x & y \end{bmatrix},$$

and in particular, for $(1,2,2)$ it is

$$A = \begin{bmatrix} -8 & 4 & 4 \\ -4 & 0 & 2 \end{bmatrix}.$$

Since $A_{(x,y)} = \begin{bmatrix} -8 & 4 \\ -4 & 0 \end{bmatrix}$ is invertible, by the Implicit Function Theorem we can solve for x, y uniquely in terms of z in a neighborhood of $(1,2,2)$, say $(x,y) = (h_1(z), h_2(z))$. This is the desired parametrization.

(b) We have $A_{(y,z)} = \begin{bmatrix} 4 & 4 \\ 0 & 2 \end{bmatrix}$ which is also invertible, so as in (a) we can solve for (y,z) in terms of x. But $A_{(x,z)} = \begin{bmatrix} -8 & 4 \\ -4 & 2 \end{bmatrix}$ is not invertible so we cannot necessarily solve for (x,z) in terms of y.