MATH 425b FINAL EXAM SOLUTIONS
SPRING 2009
Prof. Alexander

(1)(a) We have

$$\begin{aligned}\omega \wedge \omega &= f(x)g(x)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 + f(x)g(x)dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 \\ &= 2f(x)g(x)dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.\end{aligned}$$

This is $\neq 0$ if and only if there exists x where $f(x)g(x) \neq 0$, i.e. where both $f(x)$ and $g(x)$ are nonzero, since then there is a neighborhood of x where $f(x)g(x) > 0$ or where $f(x)g(x) < 0$, so for a 2-surface in that neighborhood, the integral is nonzero.

(b) If $I \cap J \neq \phi$ then clearly all 4 terms in $\omega \wedge \omega$ are 0, so suppose $I \cap J = \phi$. We have

$$\omega \wedge \omega = f(x)g(x)(dx_I \wedge dx_J + dx_J \wedge dx_I) = f(x)g(x)((-1)^\alpha + (-1)^\beta)dx_{[I,J]},$$

where α is the number of pairs (i,j) with $i \in I$, $j \in J$ and $i > j$, and β is the number of pairs (i,j) with $i \in I$, $j \in J$ and $j > i$. The total number of pairs of both types is thus $\alpha + \beta = k^2$ which is odd. Therefore one of α, β is odd and the other is even, so $(-1)^\alpha + (-1)^\beta = 0$, so $\omega \wedge \omega = 0$.

(c) ω exact means $\omega = d\xi$ for some $(k-1)$-form ξ. Then

$$d(\xi \wedge d\beta) = d\xi \wedge d\beta + (-1)^{k-1}\xi \wedge d^2\beta = \omega \wedge d\beta,$$

which shows $\omega \wedge d\beta$ is exact.

(2)(a) By Parseval, $\|f^{(k)} - f\|_2^2 = \sum_{n\in\mathbb{Z}} |c_n^{(k)} - c_n|^2$.

(b) We have $|c_n| = \lim_k |c_n^{(k)}| \leq b_n$ so $|c_n^{(k)} - c_n| \leq |c_n^{(k)}| + |c_n| \leq 2b_n$, and therefore for fixed N,

$$\sum_{n\notin[-N,N]} |c_n^{(k)} - c_n|^2 \leq 4 \sum_{n\notin[-N,N]} b_n^2.$$

Given $\epsilon > 0$ we can choose N so $4\sum_{n\notin[-N,N]} b_n^2 < \epsilon/2$. Then we can choose K so that

$$k \geq K \implies \sum_{n\in[-N,N]} |c_n^{(k)} - c_n|^2 < \frac{\epsilon}{2}.$$

Then for $k \geq K$,

$$\sum_{n\in\mathbb{Z}} |c_n^{(k)} - c_n|^2 < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

which shows, since ϵ is arbitrary, that

$$\|f^{(k)} - f\|_2^2 = \sum_{n\in\mathbb{Z}} |c_n^{(k)} - c_n|^2 \to 0, \quad \text{as } k \to \infty.$$

(3)(a) Suppose $f \in \mathrm{Lip}_a(K), g \in \mathrm{Lip}_b(K)$ for some a, b. Then for all $x, y \in K$:

$$|cf(x) - cf(y)| \le |c|ad(x,y) \quad \text{so } cf \in \mathrm{Lip}(K),$$

$$|(f+g)(x) - (f+g)(y)| \le |f(x) - f(y)| + |g(x) - g(y)| \le (a+b)d(x,y) \quad \text{so } f+g \in \mathrm{Lip}(K),$$

and since f, g are bounded due to compactness of K,

$$\begin{aligned} |(fg)(x) - (fg)(y)| &\le |f(x)g(x) - f(x)g(y)| + +f(x)g(y) - f(y)g(y)| \\ &\le \|f\|_\infty |g(x) - g(y)| + \|g\|_\infty |f(x) - f(y)| \\ &\le (b\|f\|_\infty + a\|g\|_\infty)d(x,y), \qquad \text{so } fg \in \mathrm{Lip}(K). \end{aligned}$$

(b) We need to show $F_{c,M}$ is closed, pointwise bounded and equicontinuous. If $f_n \in F_{c,M}$ and $f_n \to f$ in $C(K)$ (i.e. uniformly), then for all x, y, $|f(x)| = \lim_n |f_n(x)| \le M$ and $|f(x) - f(y)| = \lim_n |f_n(x) - f_n(y)| \le cd(x,y)$, so $f \in F_{c,M}$. Thus $F_{c,M}$ is closed. $F_{c,M}$ is uniformly bounded by M so it is pointwise bounded. Given $\epsilon > 0$, we have

$$f \in F_{c,M}, x, y \in K, d(x,y) < \frac{\epsilon}{c} \implies |f(x) - f(y)| \le cd(x,y) < \epsilon,$$

so $F_{c,M}$ is equicontinuous. This shows it is compact.

(4) For all Ω we have by Stokes Theorem and the given physical law:

$$\begin{aligned} \int_\Omega -4\pi G\rho(x)\ dx_1dx_2dx_3 &= -4\pi Gm(\Omega) \\ &= \iint_{\partial\Omega} F \cdot \mathbf{n}\, dA \\ &= \int_{\partial\Omega} \omega_{(F)} \\ &= \int_\Omega d\omega_{(F)} \\ &= \int_\Omega \mathrm{div}\ F\ dx_1dx_2dx_3. \end{aligned}$$

Since the integrands are continuous functions, if they ever differed, there would be a region Ω in which $-4\pi G\rho(x) - \mathrm{div}\, F(x)$ was always positive or always negative, so the integrals over Ω would be different. Thus in fact the integrands must be equal everywhere, that is, $-4\pi G\rho(x) = \mathrm{div}\, F(x)$ for all x.