MATH 425b ASSIGNMENT 1 SOLUTIONS
SPRING 2009
Prof. Alexander

Chapter 7

(1) Suppose $f_n \to f$ uniformly. Let $\epsilon > 0$. There exists an N such that $||f_N - f||_\infty < \epsilon$. Since f_N is bounded, there exists an M_N such that $|f_N(x)| \le M_N$ for all x. Then for all x,

$$|f(x)| \le |f_N(x)| + |f_N(x) - f(x)| < M_N + \epsilon,$$

so f is uniformly bounded.

(2) Suppose $f_n \to f$ and $g_n \to g$, both uniformly. Let $\epsilon > 0$. There exist N_1, N_2 such that

$$n \ge N_1 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2} \quad \text{for all } x,$$

$$n \ge N_2 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2} \quad \text{for all } x.$$

Then for $n \ge \max(N_1, N_2)$,

$$\big|\, (f_n(x) + g_n(x)) - (f(x) + g(x)) \,\big| \le \big|f_n(x) - f(x)\big| + \big|g_n(x) - g(x)\big| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon,$$

for all x. Thus $f_n + g_n \to f + g$ uniformly.

Now suppose also that each f_n and g_n is bounded. By Problem 1, $\{f_n\}$ and $\{g_n\}$ are uniformly bounded, that is, there exists M such that $|f_n(x)| \le M$ and $|g_n(x)| \le M$ for all x. Let $\epsilon > 0$. There exist N_3, N_4 such that

$$n \ge N_3 \implies |f_n(x) - f(x)| < \frac{\epsilon}{2M} \quad \text{for all } x,$$

$$n \ge N_4 \implies |g_n(x) - g(x)| < \frac{\epsilon}{2M} \quad \text{for all } x.$$

Then for $n \ge \max(N_3, N_4)$, for all x,

$$\begin{aligned} |f_n(x)g_n(x) - f(x)g(x)| &= |f_n(x)(g_n(x) - g(x)) + (f_n(x) - f(x))g(x)| \\ &\le |f_n(x)|\ |g_n(x) - g(x)| + |f_n(x) - f(x)|\ |g(x)| \\ &< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2M} \cdot M \\ &= \epsilon. \end{aligned}$$

Thus $f_n g_n \to fg$ uniformly.

(3) Here is one example: let $f_n(x) = f(x) = g(x) = x, g_n(x) = x - \frac{1}{n}$, for $n \geq 1$ and $x \geq 0$. Then $f_n \to f$ uniformly and $g_n \to g$ uniformly, but $\sup_{x\geq 0} |f_n(x)g_n(x) - f(x)g(x)| = \sup_{x\geq 0} |x/n| = \infty$. Thus $f_n g_n \not\to fg$ uniformly.

(6) We decompose into two series:

$$\sum_n (-1)^n \frac{x^2+n}{n^2} = \sum_n (-1)^n \frac{x^2}{n^2} + \sum_n (-1)^n \frac{1}{n}, \tag{1}$$

which is legitimate provided the two series on the right both converge. For each term in the first series on the right side of (**??**), the maximum over x necessarily occurs at an endpoint:

$$\left|(-1)^n \frac{x^2}{n^2}\right| \leq \frac{\max(|a|^2, |b|^2)}{n^2} \quad \text{for all } x \in [a,b],$$

and

$$\sum_n \frac{\max(|a|^2, |b|^2)}{n^2}$$

converges, so by the Weierstrass M-test, the first series on the right side of (**??**) converges uniformly. The second series converges by the Alternating Series Test, and the series doesn't depend on x so the convergence is necessarily uniform in x. Therefore by problem 2, the sum of the two series (the left side of (**??**)) also converges uniformly in $[a,b]$, for all $a < b$.

For a fixed x,

$$\left|(-1)^n \frac{x^2+n}{n^2}\right| = \frac{x^2+n}{n^2} \geq \frac{n}{n^2} = \frac{1}{n},$$

and $\sum_n 1/n$ diverges, so by the comparison test,

$$\sum_n \left|(-1)^n \frac{x^2+n}{n^2}\right|$$

diverges. This means the series on the left side of (**??**) does not converge absolutely for any x.

(8) Since $|c_n I(x-x_n)| \leq |c_n|$ and $\sum |c_n| < \infty$, the series defining f converges uniformly by the Weierstrass M-test (Theorem 7.10.) Letting $f_n(t) = \sum_{k=1}^n c_n I(x-x_n)$, this means that $f_n \to f$ uniformly. Let x be a point that is not one of the x_n's; then each f_n is continuous at x, that is, $\lim_{t\to x} f_n(t) = f_n(x)$. Since $f_n(x) \to f(x)$ as $n \to \infty$, Theorem 7.11 (applied with $A_n = f_n(x)$) says

$$\lim_{t\to x} f(t) = \lim_{n\to\infty} f_n(x) = f(x),$$

that is, f is continuous at x.

(9) Suppose $f_n \to f$ uniformly on E. Let $x_n, x \in E$ with $x_n \to x$. Then

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| \leq ||f_n - f||_\infty + |f(x_n) - f(x)|.$$

Now f is continuous by Theorem 7.12, so $|f(x_n) - f(x)| \to 0$ as $n \to \infty$. Hence $||f_n - f||_\infty + |f(x_n) - f(x)| \to 0$, so $f_n(x_n) \to f(x)$.

The converse is false. For example let $f_n(x) = x/n, f(x) \equiv 0$, on $[0, \infty)$. If $x_n \to x$ then $f_n(x_n) = x_n/n \to 0 = f(x)$. But f_n does not converge uniformly to f: $||f_n - f||_\infty = \infty$.

(I)(a) For fixed $x > 0$, since e^{-nx^2} decreases exponentially in n while nx increases only linearly, the product, which is $f_n(x)$, converges to 0. Since $f_n(0) = 0$ for all n, this shows that $f_n \to f \equiv 0$ pointwise on $[0, 1]$.

(b) Using $u = nx^2, du = 2nx\ dx$,

$$\int_0^1 f_n(x)\ dx = \int_0^1 nxe^{-nx^2}\ dx = \frac{1}{2}\int_0^n e^{-u}\ du \to \frac{1}{2}\int_0^\infty e^{-u}\ du = \frac{1}{2} \neq 0 = \int_0^1 f(x)\ dx.$$

(II) Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists N such that $||f_n - f||_\infty < \epsilon/3$ for all $n \geq N$. Since f_N is uniformly continuous, there exists $\delta > 0$ such that

$$x, y \in E, d(x, y) < \delta \implies |f_N(y) - f_N(x)| < \epsilon/3.$$

Then for $x, y \in E$ with $d(x, y) < \delta$ we have

$$|f(y) - f(x)| \leq |f(y) - f_N(y)| + |f_N(y) - f_N(x)| + |f_N(x) - f(x)| \leq ||f - f_N||_\infty + \frac{\epsilon}{3} + ||f - f_N||_\infty < \epsilon.$$

This shows that f is uniformly continuous.

(III) f_n is continuous, but f is not. If the convergence were uniform, the limit would have to be continuous, so the convergence in this case cannot be uniform.