

MATH 425b ASSIGNMENT 2 SOLUTIONS
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Chapter 7

(16) Suppose $\{f_n\}$ is equicontinuous on K , and $f_n \rightarrow f$ pointwise. Then $\{f_n(x)\}$, being convergent, is bounded for each x , so $\{f_n\}$ is pointwise bounded.

Suppose (to get a contradiction) that the convergence of f_n to f is not uniform: $\|f_n - f\|_\infty \not\rightarrow 0$. This means there exists $\epsilon > 0$ and a subsequence $\{f_{n_k}\}$ with $\|f_{n_k} - f\|_\infty > \epsilon$ for all k . But as a set, $\{f_{n_k}\}$ is pointwise bounded and equicontinuous (since $\{f_n\}$ is), so $\{f_{n_k}\}$ has a uniformly convergent subsequence by Theorem 7.25, say $f_{m_j} \rightarrow g$ uniformly for some g , where $\{f_{m_j}\} \subset \{f_{n_k}\}$. Since $f_{m_j} \rightarrow f$ pointwise, we must have $g = f$, so $\|f_{m_j} - f\|_\infty \rightarrow 0$. But this contradicts $\|f_{n_k} - f\|_\infty > \epsilon$ for all k . Thus $f_n \rightarrow f$ uniformly.

(18) We claim that $\{F_n\}$ is equicontinuous. Proof: By assumption there exists $M < \infty$ such that $|f_n(x)| \leq M$ for all x and n . Then for $x < y$ in $[a, b]$,

$$|F_n(y) - F_n(x)| = \left| \int_x^y f_n(t) dt \right| \leq \int_x^y |f_n(t)| dt \leq \int_x^y M dt = M|y - x|.$$

thus given $\epsilon > 0$, we have $|y - x| < \epsilon M$ implies $|F_n(y) - F_n(x)| < \epsilon$, which proves the claim. Then by Theorem 7.25, $\{F_n\}$ has a uniformly converging subsequence.

(I) Suppose \mathcal{F} is equicontinuous and $f \in \overline{\mathcal{F}}$. Then there exists a sequence $\{f_n\} \subset \mathcal{F}$ with $f_n \rightarrow f$ uniformly. Let $\epsilon > 0$. By equicontinuity, there exists $\delta > 0$ such that

$$d(x, y) < \delta \implies |f_n(y) - f_n(x)| < \epsilon \text{ for all } n \geq 1.$$

Then

$$d(x, y) < \delta \implies |f(y) - f(x)| = \lim_n |f_n(y) - f_n(x)| \leq \epsilon,$$

so the same δ “works” for f . Thus this δ “works” for all $f \in \overline{\mathcal{F}}$, so $\overline{\mathcal{F}}$ is equicontinuous.

(II)(a) Given $\epsilon > 0$ let $\delta = \epsilon^{1/\alpha}$. Then

$$f \in E, |y - x| < \delta \implies |f(y) - f(x)| \leq |y - x|^\alpha < \delta^\alpha = \epsilon,$$

i.e. this δ “works” uniformly over E . This shows E is equicontinuous.

(b) Suppose $f_n \in E$ for all n and $f_n \rightarrow f$ uniformly. Then $f(0) = \lim_n f_n(0) = 0$ and for all $x, y \in [0, 1]$,

$$|f(x) - f(y)| = \lim_n |f_n(x) - f_n(y)| \leq |y - x|^\alpha,$$

so $f \in E$.

(III)(a) Suppose $P(x) = \sum_{n=0}^N a_n x^n$, and let $\epsilon > 0$. For each $n \leq N$ there exists a rational q_n with $|a_n - q_n| < \epsilon/NA^n$. Let $Q(x) = \sum_{n=0}^N q_n x^n$. Then

$$|P(x) - Q(x)| \leq \sum_{n=0}^N |a_n - q_n| |x|^n \leq \sum_{n=0}^N \frac{\epsilon}{NA^n} A^n = \epsilon$$

for all $x \in [0, A]$, so $\|P - Q\|_\infty < \epsilon$.

(b) Let D be the set of all polynomial functions on $[0, A]$ with rational coefficients. Let $\epsilon > 0$. By the Weierstrass theorem, given $f \in C[0, A]$ there is a polynomial P with $\|f - P\|_\infty < \epsilon/2$. By (a) there is a polynomial $Q \in E$ with $\|P - Q\|_\infty < \epsilon/2$. Then $\|f - Q\|_\infty < \epsilon$, which shows D is dense in $C[0, A]$. There is a one-to-one correspondence between D and the countable set \mathbb{Q}^{N+1} (where \mathbb{Q} = rationals), by pairing $\sum_{n=0}^N q_n x^n$ with (q_0, \dots, q_N) , so D is countable.

(IV) Since $[a, b]$ is compact, each f_i is uniformly continuous. Hence given $\epsilon > 0$, for each $i \leq n$ there is a $\delta_i > 0$ such that $|y - x| < \delta_i$ implies $|f_i(y) - f_i(x)| < \epsilon/n$. Let $\delta = \min(\delta_1, \dots, \delta_n)$. Then for $f = \sum_{i=1}^n c_i f_i \in \mathcal{F}$ and $x, y \in [a, b]$ with $|y - x| < \delta$, we have

$$|f(y) - f(x)| \leq \sum_{i=1}^n |c_i| |f_i(y) - f_i(x)| < \sum_{i=1}^n 1 \cdot \frac{\epsilon}{n} = \epsilon.$$

This shows \mathcal{F} is equicontinuous.

(V)(a) Let $k \geq 1$ be the degree of P : $P(x) = a_k x^k + a_{k-1} x^{k-1} + \dots$ with $a_k \neq 0$. Then

$$\frac{P(x)}{x^k} = a_k + a_{k-1} x^{-1} + \dots \rightarrow a_k \quad \text{as } x \rightarrow \infty,$$

so for large x , $\left| \frac{P(x)}{x^k} \right| \geq \frac{1}{2} |a_k|$ and therefore $|P(x)| \geq \frac{1}{2} |a_k| x^k \rightarrow \infty$ as $x \rightarrow \infty$. Thus $\|P\|_\infty = \infty$.

(b) For all $n \neq m$ we have

$$\|P_n - P_m\|_\infty \leq \|P_n - f\|_\infty + \|f - P_m\|_\infty,$$

which is finite if n, m are large (since $\|P_n - f\|_\infty \rightarrow 0$), so by (a), $P_n - P_m$ must be a constant, call it c_{mn} . But then for fixed m and x we get

$$\lim_n c_{mn} = \lim_n (P_n(x) - P_m(x)) = f(x) - P_m(x),$$

so $\lim_n c_{mn}$ is some finite c and $f(x) = P_m(x) + c$, meaning f is a polynomial.