

MATH 425b ASSIGNMENT 3 SOLUTIONS
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Chapter 7

(20) Let $\epsilon > 0$. There exists a polynomial with $\|P - f\| < \epsilon$ (sup norm), say $P(x) = \sum_{n=0}^N c_n x^n$. f is bounded since it is continuous on the compact set $[0, 1]$, so there exists M such that $|f(x)| \leq M$ for all x . Therefore

$$\int_0^1 f(x)P(x) dx = \sum_{n=0}^N c_n \int_0^1 f(x)x^n dx = 0$$

and

$$\begin{aligned} 0 \leq \int_0^1 f(x)^2 dx &= \left| \int_0^1 f(x)^2 dx - \int_0^1 f(x)P(x) dx \right| \\ &= \left| \int_0^1 f(x)(f(x) - P(x)) dx \right| \\ &\leq \int_0^1 |f(x)||f(x) - P(x)| dx \\ &\leq \int_0^1 M\epsilon dx \\ &= M\epsilon. \end{aligned}$$

Since ϵ is arbitrary, this shows $\int_0^1 f(x)^2 dx = 0$. By Exercise 2 of chapter 6, this means $f(x)^2 = 0$ for all x , so $f(x) = 0$ for all x .

(21) The constant function $f(e^{i\theta}) \equiv 1$ for all θ is in \mathcal{A} , and vanishes nowhere, so \mathcal{A} vanishes at no point of K . The identity function $f(e^{i\theta}) = e^{i\theta}$ is in \mathcal{A} , and is one-to-one, so \mathcal{A} separates points.

To prove Rudin's hint, for any function $f(e^{i\theta}) = \sum_{n=0}^N c_n e^{in\theta}$ in \mathcal{A} we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta} d\theta = \sum_{n=0}^N c_n \int_0^{2\pi} e^{i(n+1)\theta} d\theta = 0. \tag{1}$$

For $f \in \overline{\mathcal{A}}$ there exists a sequence $\{f_n\} \subset \mathcal{A}$ with $f_n \rightarrow f$ uniformly. Hence applying (3) to

f_n ,

$$\begin{aligned}
\left| \int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta \right| &= \left| \int_0^{2\pi} (f(e^{i\theta}) - f_n(e^{i\theta})) e^{i\theta} d\theta \right| \\
&\leq \int_0^{2\pi} |f(e^{i\theta}) - f_n(e^{i\theta})| |e^{i\theta}| d\theta \\
&\leq 2\pi \|f - f_n\| \\
&\rightarrow 0 \quad \text{as } n \rightarrow \infty,
\end{aligned} \tag{2}$$

so we must have $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 0$, for all $f \in \overline{\mathcal{A}}$. But for the particular choice $f(e^{i\theta}) = e^{-i\theta}$ we have $\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} d\theta = 2\pi$, so $f \notin \overline{\mathcal{A}}$, though f is continuous on K .

Chapter 8

(1) We claim that for each n there is a polynomial P_n such that

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-1/x^2}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

To prove this, note it is true for $n = 0$, with $P_n \equiv 1$. Suppose it is true for some n . Then for $x \neq 0$,

$$\begin{aligned}
f^{(n+1)}(x) &= P_n\left(\frac{1}{x}\right) \cdot \frac{2}{x^3} e^{-1/x^2} - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right) e^{-1/x^2} \\
&= \left(\frac{2}{x^3} P_n\left(\frac{1}{x}\right) - \frac{1}{x^2} P_n'\left(\frac{1}{x}\right) \right) e^{-1/x^2}.
\end{aligned} \tag{3}$$

Letting $P_{n+1}(x)$ be the coefficient of e^{-1/x^2} on the right side of (3), we see that since P_n and P_n' are both polynomials, so is P_{n+1} . Thus the claim follows for $x \neq 0$ by induction.

Given $x > 0$, by the Mean Value Theorem there exists $\xi \in (0, x)$ such that

$$\frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = f^{(n+1)}(\xi). \tag{4}$$

The same is true for $x < 0$, with $\xi \in (x, 0)$. Now $u^k e^{-u^2} \rightarrow 0$ as $u \rightarrow \infty$, for all $k \geq 0$, so $P_{n+1}(u) e^{-u^2} \rightarrow 0$ as $u \rightarrow \infty$, so $P_{n+1}\left(\frac{1}{x}\right) e^{-1/x^2} \rightarrow 0$ as $x \rightarrow 0$. Therefore $f^{(n+1)}(\xi) \rightarrow 0$ as $\xi \rightarrow 0$ (therefore also as $x \rightarrow 0$, since $\xi \in (0, x)$), so by (4) we have $f^{(n+1)}(0) = 0$. This completes the proof of the claim for $n + 1$, so it holds for all n by induction.

(4)(a) Let $f(x) = b^x = e^{(\log b)x}$, so $f'(x) = (\log b)e^{(\log b)x}$. Then

$$\lim_{x \rightarrow 0} \frac{b^x - 1}{x} = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = \log b.$$

(b) Use L'Hospital's Rule:

$$\lim_{x \rightarrow 0} \frac{\log(1+x)}{x} = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x}}{1} = 1.$$

(c) Use (b):

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = \lim_{x \rightarrow 0} e^{\frac{\log(1+x)}{x}} = e^1 = e.$$

(d) By (c), $\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^{n/x} = e$. Since y^x is a continuous function of y , this shows

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = \lim_{n \rightarrow \infty} \left(\left(1 + \frac{x}{n}\right)^{n/x} \right)^x = e^x.$$

(5)(a) Let $g(x) = \frac{\log(1+x)}{x}$ for $x \neq 0$, so $(1+x)^{1/x} = e^{g(x)}$, and by 4b, $g(x) \rightarrow 1$ as $x \rightarrow 0$. Hence defining $g(0) = 1$ makes g continuous at 0. Using L'Hospital's Rule,

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} &= \lim_{x \rightarrow 0} \frac{\log(1+x) - x}{x^2} \\ &= \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - 1}{2x} \\ &= \lim_{x \rightarrow 0} -\frac{(1+x^2)^{-2}}{2} \\ &= -\frac{1}{2}, \end{aligned}$$

so g is also differentiable at 0. Thus we can use the chain rule:

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{e - (1+x)^{1/x}}{x} &= \lim_{x \rightarrow 0} \frac{e^{g(0)} - e^{g(x)}}{x - 0} \\ &= -\frac{d}{dx} e^{g(x)} \Big|_{x=0} \\ &= -g'(0)e^{g(0)} \\ &= \frac{e}{2}. \end{aligned}$$

(b) Let $f(x) = (e^x - 1)/x$, so we can express the quantity for which we want the limit as

$$f\left(\frac{\log n}{n}\right) = \frac{n}{\log n} (n^{1/n} - 1).$$

Letting $g(x) = e^x$, we have

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0} = g'(0) = 1,$$

so the desired limit is

$$\lim_{n \rightarrow \infty} f\left(\frac{\log n}{n}\right) = 1.$$

(6)(a) Taking $x = y = 0$ shows $f(0)^2 = f(0)$ so $f(0) = 0$ or 1 for all x . But $f(x) = f(x+0) = f(x)f(0)$ so if $f(0) = 0$ then $f(x)$ would be 0 for all x . Therefore $f(0) = 1$.

Let $g(x) = \log f(x)$, so $g(0) = 0$ and $g(x+y) = g(x) + g(y)$. Then

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{g(h)}{h} = \lim_{h \rightarrow 0} \frac{g(h) - g(0)}{h} = g'(0) \quad \text{for all } x.$$

Letting $c = g'(0)$, this shows that $g(x) = cx + c'$ for some c' . Since $g(0) = 0$ we must have $c' = 0$, so $g(x) = cx$, which means $f(x) = e^{cx}$.

(b) Rudin is a bit unclear—we still assume f is not 0 , we just replace differentiability with continuity.

Since $g(x+y) = g(x) + g(y)$, taking $x = y$ shows $g(2x) = 2g(x)$, and then by easy induction on m ,

$$g(mx) = g((m-1)x + x) = g((m-1)x) + g(x) = (m-1)g(x) + g(x) = mg(x),$$

for all m and x . Hence also

$$g(x) = g\left(n \cdot \frac{x}{n}\right) = ng\left(\frac{x}{n}\right)$$

for all n, x , so $g\left(\frac{x}{n}\right) = \frac{1}{n}g(x)$. Therefore for all m, n ,

$$g\left(\frac{m}{n}\right) = g\left(m \cdot \frac{1}{n}\right) = mg\left(\frac{1}{n}\right) = mg\left(\frac{1}{n} \cdot 1\right) = \frac{m}{n}g(1).$$

Letting $a = g(1)$ we thus have $g(x) = ax$ for all rational x . Since g is continuous, for irrational x we can take a sequence of rationals $x_k \rightarrow x$ and

$$g(x) = \lim_k g(x_k) = \lim_k ax_k = ax.$$

Thus $f(x) = e^{ax}$.

(A) ((a) \implies (b)) Suppose $\sum_{n=0}^{\infty} a_n$ converges. Then the radius of convergence is at least 1 , so f is defined at least on $[0, 1]$. For $x \in [0, 1]$ we have

$$\left|f(x) - \sum_{n=1}^N a_n x^n\right| \leq \sum_{n=N+1}^{\infty} |a_n| |x|^n \leq \sum_{n=N+1}^{\infty} |a_n|.$$

The last sum does not depend on x , and approaches 0 as $N \rightarrow \infty$. Thus the series converges uniformly to $f(x)$ on $[0, 1]$.

((b) \implies (c)) Suppose $\sum_{n=1}^{\infty} a_n x^n$ converges uniformly on $[0, 1]$. Then the limit $f(x)$ is a continuous function, so f is bounded on $[0, 1]$, hence also on $[0, 1)$.

((c) \implies (a)) Suppose $\sum_{n=0}^{\infty} a_n = \infty$. Given $M > 0$ there exists N such that $\sum_{n=0}^N a_n > M$. Then for x sufficiently close to 1 we have $f(x) \geq \sum_{n=0}^N a_n x^n > M$. This shows that f is unbounded on $[0, 1]$.

(B) Suppose $g(x) \neq g(y)$ for some $x < y$. Let $\epsilon, \delta > 0$; we will show that δ does not “work” for this ϵ for some multiple of g . Essentially this is because if you multiply g by a big enough constant c , the increments of g become very large. Specifically, let $x = x_0 < x_1 < \dots < x_n = y$ with $x_i - x_{i-1} < \delta$ for all $1 \leq i \leq n$. If δ “works” for some positive multiple cg of g , then

$$c|g(y) - g(x)| = |cg(y) - cg(x)| < c \left| \sum_{i=1}^n (g(x_i) - g(x_{i-1})) \right| \leq \sum_{i=1}^n |cg(x_i) - cg(x_{i-1})| \leq n\epsilon.$$

But for large c this is false, because $c|g(y) - g(x)| > n\epsilon$. In other words, δ does not “work” for all multiples cg of g . Thus \mathcal{A} is not equicontinuous.

(C)(a) Since f is never 0, \mathcal{A}_1 vanishes at no point of $[0, 1]$. If $(x_1, y_1) \neq (x_2, y_2)$ then either $x_1 \neq x_2$ or $y_1 \neq y_2$. If $x_1 \neq x_2$ then $g(x_1, y_1) \neq g(x_2, y_2)$. If $y_1 \neq y_2$ then $f(x_1, y_1) \neq g(x_2, y_2)$. This shows that \mathcal{A}_1 separates points. By the Stone-Weierstrass Theorem, the uniform closure of \mathcal{A}_1 is all of $C([0, 1]^2)$, so in particular it includes h .

(b) Every polynomial of form $c + (x - \frac{1}{2})^2 R(x)$, with R a polynomial and c a constant, is in \mathcal{A}_2 . In particular the strictly increasing function $(x - \frac{1}{2})^3 \in \mathcal{A}_2$, which shows that \mathcal{A}_2 separates points. Taking $c > 0$ and $R \equiv 1$ we see that \mathcal{A}_2 vanishes at no point. By the Stone-Weierstrass Theorem, \mathcal{A}_2 is dense in $C[0, 1]$.