MATH 425b ASSIGNMENT 4 SOLUTIONS
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Chapter 8:

(12)(a) For $n \neq 0$,

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\ dx \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta}(\cos nx + i\sin nx)\ dx \\ &= \frac{1}{2\pi}\int_{-\delta}^{\delta}\cos nx\ dx \quad \text{(since } \sin nx \text{ is odd)} \\ &= \frac{1}{2\pi n}\sin nx\Big|_{-\delta}^{\delta} \\ &= \frac{\sin n\delta}{n\pi}, \end{aligned}$$

and

$$c_0 = \frac{1}{2\pi}\int_{-\delta}^{\delta} 1\ dx = \frac{\delta}{\pi}.$$

(b) The series converges pointwise except possibly at the discontinuity at $x = \pm\delta$, so

$$1 = f(0) = \sum_{n=-\infty}^{\infty} c_n = \frac{\delta}{\pi} + 2\sum_{n=1}^{\infty}\frac{\sin n\delta}{\pi n}.$$

Rearranging this gives

$$\frac{\pi - \delta}{2} = \sum_{n=1}^{\infty}\frac{\sin n\delta}{n}.$$

(c) We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\ dx = \frac{1}{2\pi}\int_{-\delta}\delta\ dx = \frac{\delta}{\pi}$$

and

$$\sum_{n=-\infty}^{\infty}|c_n|^2 = \frac{\delta^2}{\pi^2} + 2\sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{\pi^2 n^2}.$$

By Parseval's Theorem these are equal. Multiplying each one by π^2/δ we get

$$\pi = \delta + 2\sum_{n=1}^{\infty}\frac{\sin^2 n\delta}{\pi^2 n^2},$$

so

$$\frac{\pi - \delta}{2} = \sum_{n=1}^{\infty} \frac{\sin^2 n\delta}{\pi^2 n^2}.$$

(e) $\sin^2 n\pi/2$ is 1 for n odd, 0 for n even so taking $\delta = \pi/2$ in (c) we get

$$\frac{\pi}{4} = \frac{\pi - \frac{\pi}{2}}{2} = \sum_{n=1}^{\infty} \frac{\sin^2 \frac{n\pi}{2}}{n^2 \frac{\pi}{2}} = \frac{2}{\pi} \sum_{n \text{ odd}} \frac{1}{n^2}.$$

Hence

$$\sum_{n \text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

(13) Let $f(x) = x$ for $0 \le x \le 2\pi$. Using integration by parts, for $n \neq 0$ we get the Fourier coefficient

$$\begin{aligned} c_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx}\, dx &= \frac{1}{2\pi} \left(\int_0^{2\pi} x \cos nx\, dx - i \int_0^{2\pi} x \sin nx\, dx \right) \\ &= \frac{1}{2\pi} \left(0 + \frac{2\pi}{n} i \right) \\ &= \frac{i}{n}, \end{aligned}$$

while

$$c_0 = \frac{1}{2\pi} \int_0^{2\pi} x\, dx = \pi.$$

Hence

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2}$$

while

$$\frac{1}{2\pi} \int_0^{2\pi} f(x)^2\, dx = \frac{1}{2\pi} \frac{x^3}{3} \bigg|_0^{2\pi} = \frac{4\pi^2}{3}.$$

By Parseval's identity these are equal, that is,

$$\pi^2 + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4\pi^2}{3}.$$

Therefore $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

(14) Let $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$. The Fourier coefficients of f are

$$c_n = \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} f(x) \cos nx\, dx - i \int_{-\pi}^{\pi} f(x) \sin nx\, dx \right).$$

Since f is even, the second integral is 0, and the first is $\frac{1}{\pi}\int_0^\pi (\pi - x)^2 \cos nx \; dx$. Multiplying out $(\pi - x)^2$ and then doing integration by parts we get $c_n = 2/n^2$ for $n \neq 0$. For $n = 0$ we have by routine calculation

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) \; dx = \frac{\pi^2}{3}.$$

Therefore

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} c_n \cos nx + i \sum_{n=-\infty}^{\infty} c_n \sin nx,$$

and by Theorem 8.14, the convergence is pointwise except perhaps at the endpoints $\pm\pi$. Since f is real-valued, the last sum must be 0, so

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \cos nx = c_0 + \sum_{n=1}^{\infty} (c_n + c_{-n}) \cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Since the convergence is pointwise, we can plug in $x = 0$ to get

$$\pi^2 = f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2},$$

which becomes $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$.

Next, routine calculation shows

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 \; dx = \frac{\pi^4}{5},$$

while

$$\sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{\pi^4}{9} + 2\sum_{n=1}^{\infty} \frac{4}{n^4}.$$

By Parseval's identity these are equal, which becomes

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

(15) From p. 189 below (77), we have

$$D_N(x) = \frac{e^{i(N+1)x} - e^{-iNx}}{e^{ix} - 1}, \quad x \neq 0.$$

Therefore by summing finite geometric series we get

$$\begin{aligned}
\sum_{n=0}^{N} D_n(x) &= \frac{1}{e^{ix}-1}\left(\sum_{n=0}^{N} e^{i(N+1)x} - \sum_{n=0}^{N} e^{-iNx}\right) \\
&= \frac{1}{e^{ix}-1}\left(e^{ix}\frac{1-e^{i(N+1)x}}{1-e^{ix}} - \frac{1-e^{-i(N+1)x}}{1-e^{-ix}}\right) \\
&= \frac{1}{e^{ix}-1}\left(\frac{1-e^{i(N+1)x}}{e^{-ix}-1} + \frac{1-e^{-i(N+1)x}}{e^{-ix}-1}\right) \\
&= \frac{2-2\cos(N+1)x}{(e^{ix}-1)(e^{-ix}-1)} \\
&= \frac{2-2\cos(N+1)x}{2-e^{ix}-e^{-ix}} \\
&= \frac{2-2\cos(N+1)x}{2-2\cos x} \\
&= \frac{1-\cos(N+1)x}{1-\cos x},
\end{aligned}$$

so

$$K_N(x) = \frac{1}{N+1}\sum_{n=0}^{N} D_n(x) = \frac{1}{N+1}\frac{1-\cos(N+1)x}{1-\cos x}, \quad x \neq 0.$$

(a) By periodicity we need only consider $x \in [-\pi, \pi]$. For $x \neq 0$ we have $\cos x < 1$ and $\cos(N+1)x \leq 1$, so from the representation above, $K_N(x) \geq 0$. For $x = 0$, since $D_N(x) = 2N+1$ we have $K_N(x) \geq 0$.

(b) By the equation after (81), $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\ dx = 1$ for all n, so

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)\ dx = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)\ dx = \frac{1}{N+1}(N+1) = 1.$$

(c) Suppose $\delta > 0$ and $\delta \leq |x| \leq \pi$. Since $\cos x$ is decreasing on $[0, \pi]$, and $\cos x = \cos(-x)$, we have $1-\cos x \geq 1-\cos\delta$. Also $1-\cos(N+1)x \leq 2$, so

$$K_N(x) \leq \frac{1}{N+1}\ \frac{2}{1-\cos\delta}.$$

This completes (c).

We now prove the equality involving $\sigma_N(f;x)$. Equation (78) says

$$s_N(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_N(t)\ dt,$$

so indeed

$$\begin{aligned}\sigma_N(f;x) &= \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;x) \qquad (1)\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{N+1}\sum_{n=0}^{N} D_n(t)\ dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)\ dt.\end{aligned}$$

Now suppose f is continuous with period 2π. Then f is bounded, say $|f(x)| \le M$ for all x. Let $\epsilon > 0$. f is uniformly continuous so there exists $\delta > 0$ such that

$$|t| < \delta \quad \text{implies } |f(x-t) - f(x)| < \epsilon, \quad \text{for all } x, t.$$

Now by (b) and (1),

$$\begin{aligned}|\sigma_N(f;x) - f(x)| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t) - f(x))K_N(t)\ dt\right|\\ &\le \frac{1}{2\pi}\int_{-\delta}^{\delta} |f(x-t) - f(x)|\ K_N(t)\ dt\\ &\quad + \frac{1}{2\pi}\int_{\delta}^{\pi} |f(x-t) - f(x)|\ K_N(t)\ dt\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{-\delta} |f(x-t) - f(x)|\ K_N(t)\ dt\\ &= (I) + (II) + (III).\end{aligned}$$

We bound these three terms separately. By (a) and (b),

$$(I) \le \frac{1}{2\pi}\int_{-\delta}^{\delta} \epsilon\ K_N(t)\ dt \le \epsilon \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(t)\ dt = \epsilon.$$

By (c), if N is large enough,

$$(II) \le \frac{1}{2\pi}\int_{\delta}^{\pi} 2M\ \frac{1}{N+1}\ \frac{2}{1-\cos\delta}\ dt < \epsilon.$$

and similarly

$$(III) < \epsilon.$$

Hence (I) + (II) + (III) $< 3\epsilon$ if N is large. ϵ is arbitrary and the bound is uniform in x, so this shows $\sigma_N(f;x) \to f(x)$ uniformly in x.

(19) Suppose f is continuous with period 2π and α/π is irrational. Let $g_k(x) = e^{ikx}$. Consider first $k \neq 0$. Summing a finite geometric series we get that

$$\frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha) = \frac{1}{N}\sum_{n=1}^{N} e^{ikx}e^{ikn\alpha} = \frac{1}{N}e^{ikx}\frac{1-e^{ik(N+1)\alpha}}{1-e^{ik\alpha}}. \tag{2}$$

Since α/π is irrational, we have for $k \neq 0$ that $e^{ik\alpha} = e^{i\pi(k\alpha/\pi)} \neq 1$, since $k\alpha/\pi$ is not an integer. Therefore the last denominator in (2) is not 0, so

$$\lim_N \frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha) = \lim_N \frac{1}{N}e^{ikx}\frac{1-e^{ik(N+1)\alpha}}{1-e^{ik\alpha}} = 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_k(t)\ dt.$$

Next, for $k=0$ we have $g_k \equiv 1$ so

$$\lim_N \frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha) = \lim_N \frac{1}{N}\cdot N = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} g_k(t)\ dt.$$

By Stone-Weierstrass, given $\epsilon > 0$ there exists a trigonometric polynomial

$$\varphi(x) = \sum_{k=-K}^{K} c_k g_k(x)$$

with $||f-\varphi||_\infty < \epsilon$. By the above results for g_k, we have

$$\begin{aligned}\lim_N \frac{1}{N}\sum_{n=1}^{N} \varphi(x+n\alpha) &= \sum_{k=-K}^{K} c_k \lim_N \frac{1}{N}\sum_{n=1}^{N} g_k(x+n\alpha)\\ &= \sum_{k=-K}^{K} c_k \frac{1}{2\pi}\int_{-\pi}^{\pi} g_k(t)\ dt\\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(t)\ dt. \end{aligned} \tag{3}$$

In other words, the desired result is true for trigonometric polynomials like φ. To prove it for f, we compare the expressions for f and φ:

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx &= \frac{1}{N}\sum_{n=1}^{N}\varphi(x+n\alpha) - \frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(x)\ dx\\ &\quad + \frac{1}{N}\sum_{n=1}^{N}[f(x+n\alpha)-\varphi(x+n\alpha)]\\ &\quad + \frac{1}{2\pi}\int_{-\pi}^{\pi}[\varphi(x)-f(x)]\ dx\\ &= (I) + (II) + (III).\end{aligned}$$

By (3), we have (I) $\to 0$ as $N \to \infty$, so $|(I)| < \epsilon$ if N is large. Also, since $||f - \varphi||_\infty < \epsilon$,

$$|(II)| \leq \frac{1}{N} \sum_{n=1}^{N} \epsilon = \epsilon \quad \text{and} \quad |(III)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \epsilon \, dx = \epsilon,$$

so (I) + (II) + (III) $< 3\epsilon$. Since ϵ is arbitrary, this proves the desired result.

Chapter 9:

(3) If $x \neq y$ then $Ax - Ay = A(x - y) \neq 0$, that is, $Ax \neq Ay$. This shows A is one-to-one.

(4) Let $T \in L(X, Y)$, and let $z \in Z = \text{range}(T) = \{y \in Y : y = Tx \text{ for some } x \in X\}$. We show Z is a vector space: suppose $u, v \in Z$ and c is a scalar. Then $u = Tx, v = Tw$ for some $x, w \in X$, and $T(cx + w) = cTx + Tw = cu + v$, so $cu + v \in Z$. Thus Z is a vector space.

Next let $N = \{x \in X : Tx = 0\}$ be the nullspace. Suppose $g, h \in N$ and c is a scalar. Then $T(cg + h) = cTg + Th = c \cdot 0 + 0 = 0$, so $cg + h \in N$. Thus N is a vector space.

(5) Let $A \in L(\mathbb{R}^n, \mathbb{R})$, let $\{e_1, .., e_n\}$ be the standard basis, let $y_i = Ae_i \in \mathbb{R}$, and let $\mathbf{y} = (y_1, .., y_n)$. The for every $\mathbf{x} \in \mathbb{R}^n$,

$$A\mathbf{x} = A(\sum_{i=1}^{n} x_i e_i) = \sum_{i=1}^{n} x_i A e_i = \sum_{i=1}^{n} x_i y_i = \mathbf{x} \cdot \mathbf{y}.$$

By the Schwarz inequality, for all $x \neq 0$, $|A\mathbf{x}| = |\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}| \, |\mathbf{y}|$, so

$$\sup_{x \neq 0} \frac{|A\mathbf{x}|}{|\mathbf{x}|} \leq |\mathbf{y}|,$$

meaning $||A|| \leq |y|$. To show equality actually holds, consider two cases: if $y = 0$, then $A = 0$, so $||A|| = |\mathbf{y}|$. If $y \neq 0$ then

$$||A|| = \sup_{x \neq 0} \frac{|A\mathbf{x}|}{|\mathbf{x}|} \geq \frac{|A\mathbf{y}|}{|\mathbf{y}|} = \frac{|\mathbf{y}|^2}{|\mathbf{y}|} = |\mathbf{y}|,$$

so again $||A|| = |\mathbf{y}|$.