MATH 425b ASSIGNMENT 6
SPRING 2009
Prof. Alexander
Due Monday March 30.

Chapter 9 #16, 23, 28 and:

(I) Show that $Tx = \sin x$ is not a contraction on $[-1, 1]$.

(II)(a) Give an example of a differentiable map $f : \mathbb{R} \to \mathbb{R}$ whose fixed points are exactly the integers.

(b) Show that such a map must have points x where $|f'(x)| > 1$.

(III) Let $f(x) = (\sqrt{2})^x$ for $x \in \mathbb{R}$. Given $x_0 \in \mathbb{R}$, define a sequence by $x_{n+1} = f(x_n)$ for $n \geq 0$.

(a) Roughly sketch $y = f(x)$ and $y = x$ on the same graph.

(b) Find all fixed points of f.

(c) Show that $\{x_n\}$ is monotone.

(d) For which $x_0 \in \mathbb{R}$ will $\{x_n\}$ converge, and to what limit?

(IV) Suppose $f : (a, b) \times (c, d) \to \mathbb{R}$ is twice continuously differentiable, $\frac{\partial^2 f}{\partial y^2}(x, y) < 0$ for all x, y, and for each fixed x there is a value of y, call it $g(x)$, where $f(x, y)$ is maximized over y. Show that g is continuously differentiable.

(V) The expression in the Implicit Function Theorem for the derivative of g ((58) page 225) can be viewed as a relation between increments: if we move $\mathbf{y}$ from $\mathbf{b}$ to $\mathbf{b} + \Delta\mathbf{y}$, then the corresponding $\mathbf{x} = g(\mathbf{y})$ moves from $\mathbf{a}$ to $\mathbf{a} + \Delta\mathbf{x}$, and $g'(\mathbf{b})$ gives the approximate relation between $\Delta\mathbf{x}$ and $\Delta\mathbf{y}$. Use this idea to solve the following.

(a) Let $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{x} = (x_1, x_2)$, and

$$f_1(\mathbf{x}, \mathbf{y}) = y_1^2 + y_2^2 + y_3^2 - x_1^2 + x_2^2 - 1, \quad f_2(\mathbf{x}, \mathbf{y}) = y_1^2 - y_2^2 + y_3^2 + x_1^2 + 2x_2^2 - 21.$$

The point $(3, 2, 1, 1, 2)$ is then on the surface given by $(f_1, f_2) = (0, 0)$. Suppose we move $\mathbf{y}$ away from $(1, 1, 2)$ in the direction $(0, 1, 1)$. What direction must $\mathbf{x}$ move away from $(3, 2)$, to keep $(\mathbf{x}, \mathbf{y})$ on the surface $(f_1, f_2) = (0, 0)$? (Here when we refer to moving in a direction, we mean the instantaneous direction we start out in—if we move $\mathbf{x}$ and $\mathbf{y}$ in straight lines for a positive amount of time, then we will leave the surface because the surface is curved.)

(b) Suppose instead that we move $\mathbf{x}$ away from $(3, 2)$ in the direction of some vector $\mathbf{h} = (h_1, h_2)$. For which $\mathbf{h}$ does there exist a $\mathbf{k} = (k_1, k_2, k_3)$ such that we can then stay on the surface $(f_1, f_2) = (0, 0)$ by also moving $\mathbf{y}$ in direction $\mathbf{k}$?

(c) We now change the function slightly:

$$f_1(\mathbf{x}, \mathbf{y}) = y_1^2 + y_2^2 + y_3^2 - x_1^2 + x_2^2, \quad f_2(\mathbf{x}, \mathbf{y}) = y_1^2 - y_2^2 + y_3^2 + x_1^2 + 2x_2^2 - 22,$$

and change the starting point to $(3, 2, 1, 0, 2)$. Redo part (b) for this situation.

HINTS:

(16) On what intervals is the derivative positive and negative? What does this say about f being one to one or not?

(I) Use the fact that $T'(0) = 1$.

(II)(a) Think graphically–a fixed point is where the graph of f crosses what line?

(III)(b) What property (suggested by your graph in (a)) of f limits the number of times its graph can cross a straight line? Use the Mean Value Theorem to formalize the argument. The only fixed points are small integers, so you can easily find them, but the main point is to prove there aren't more.

(c) For what values of x is $f(x) < x$? Use monotonicity of $f(x) - x$.

(d) Convergence follows from monotonicity and (what?) Recall from lecture that any limit x of such a sequence $\{x_n\}$ must satisfy $f(x) = x$.

(IV) You have to think about how to turn this into an application of one of the main theorems of Chapter 9. What equation is satisfied where a function is maximized?