MATH 425b ASSIGNMENT 6 SOLUTIONS
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Chapter 9:

(16) $f(t) = t + 2t^2 \sin 1/t$. For $t \neq 0$, $f'(t) = 1 + 4t \sin \frac{1}{t} - 2\cos\frac{1}{t}$, so for $t \in (-1, 1)$, $|f'(t)| \leq 1 + 4 + 2 = 7$. For $t = 0$,

$$\lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \left(1 + 2t \sin \frac{1}{t}\right) = 1,$$

so $f'(0) = 1$. Thus $|f'(t)| \leq 7$ for all $t \in (-1, 1)$. Also

$$|t| < \frac{1}{8} \quad \text{implies} \quad \left|4t \sin \frac{1}{t}\right| \leq \frac{1}{2}. \tag{1}$$

For $t = 1/n\pi$, n even, with $|t| < 1/8$ we have $1 - 2\cos 1/t = -1$ so $f'(t) < 0$. For $t = 1/n\pi$, n odd, with $t < 1/8$ we have $1 - 2\cos 1/t = 3$ so $f'(t) > 0$, using (1). Thus as $t \to 0$, f alternates between intervals where $f' > 0$ and $f' < 0$ (i.e. where f increases and decreases) so f is not one-to-one.

(23) $f(x, y_1, y_2) = x^2 y_1 + e^x + y_2$. Clearly $f(0, 1, -1) = 0$. We have $(D_1 f)(x, y_1, y_2) = 2xy_1 + e^x$ so $(D_1 f)(0, 1, -1) = 1 \neq 0$. In the notation of the Implicit Function Theorem we have $A_x = (D_1 f)(x, y_1, y_2)$ (which is 1×1, i.e. a scalar.) Hence in a neighborhood of $(0, 1, -1)$ we can solve $f(x, y_1, y_2) = 0$ for x in terms of $\mathbf{y} = (y_1, y_2)$, obtaining $x = g(y_1, y_2)$, so that $f(g(y_1, y_2), y_1, y_2) = 0$. By the Implicit Function Theorem we have $g'(1, -1) = -A_x^{-1} A_y$, where $A_x = (D_1 f)(0, 1, -1) = 1$ and A_y is the 1×2 matrix

$$A_y = [(D_2 f)(0, 1, -1) \;\; (D_3 f)(0, 1, -1)] = [x^2 \;\; 1] = [0 \;\; 1],$$

so $g'(1, -1) = [0 \;\; -1]$. The entries of this matrix are $(D_1 g)(1, -1) = 0$, $(D_2 g)(1, -1) = -1$.

(28) φ is given by three different formulas: $\varphi_1(x, t) = x$, $\varphi_2(x, t) = -x + 2\sqrt{t}$, $\varphi_3(x, t) = 0$. φ is clearly continuous in the open regions where a single formula applies, that is for $0 < x < \sqrt{t}$ or for $\sqrt{t} < x < 2\sqrt{t}$ or for $x > 2\sqrt{t}$. There is a parabola $x = \sqrt{t}$ in the (x, t) plane separating the region where $\varphi = \varphi_1$ from the region $\varphi = \varphi_2$, and a second lower parabola separating $\varphi = \varphi_2$ from $\varphi = \varphi_3$. The positive t axis separates $\varphi = \varphi_3$ from $\varphi = \varphi_1$. The values defined for φ on the first parabola satisfy $\varphi = \varphi_1 = \varphi_2$ on that parabola, and it follows from this that φ is continuous at all points on the parabola. The same goes for the other parabola and the positive t axis. Thus φ is continuous at all points.

In the region below the lower parabola in the upper right quadrant and in the entire upper left quadrant, $\varphi = \varphi_3 = 0$ is constant, and it follows that $(D_2\varphi)(x,0) = 0$ for all $x \in \mathbb{R}$.

Now for fixed t the graph of $\varphi(x,t)$ as a function of x is a triangle, and $f(t)$ is the area under it so long as $2\sqrt{t} < 1$, i.e. $0 \le t < 1/4$, so for such t, $f(t) = \frac{1}{2} \cdot 2\sqrt{t} \cdot \sqrt{t} = t$. If $-1/4 < t < 0$ then $f(t)$ is the negative of the area of a similar triangle, so $f(t) = -|t| = t$. It follows that $f'(0) = 1$, but $\int_{-1}^{1} (D_2\varphi)(x,0)\, dx = 0$.

(I) $T'(0) = \cos 0 = 1$, so given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\begin{aligned} |x| < \delta &\implies \left| \frac{\sin x - \sin 0}{x - 0} - 1 \right| < \epsilon \\ &\implies \left| \frac{\sin x - \sin 0}{x - 0} \right| > 1 - \epsilon \\ &\implies |\sin x - \sin 0| > (1-\epsilon)|x - 0| \\ &\implies |T(x) - T(0)| > (1-\epsilon)|x - 0|. \end{aligned}$$

Thus there is no $c < 1$ such that $T(y) - T(x) < c|y - x|$ for all x, y, which means T is not a contraction.

(II)(a) For example, let $f(x) = x + c \sin \pi x$ for some $c > 0$. Then $f(x) = x \iff \sin \pi x = 0 \iff x \in \mathbb{Z}$.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable with $\mathbb{Z} = \{\text{fixed points of } T\}$. Let $g(x) = f(x) - x$. Then $g(0) = g(1) = 0$ and (since there are no fixed points of f in $(0,1)$) g is not constant on $[0,1]$, so there exists $z \in (0,1)$ where $g(z) \ne 0$.

If $g(z) > 0$ then by the Mean Value Theorem there exists $\xi \in (0,z)$ where

$$g'(\xi) = \frac{g(z) - g(0)}{z - 0} = \frac{g(z)}{z} > 0.$$

But then $f'(\xi) = g'(\xi) + 1 > 1$.

Alternatively, if $g(z) < 0$ then by the Mean Value Theorem there exists $\eta \in (z,1)$ where

$$g'(\eta) = \frac{g(1) - g(z)}{1 - z} = -\frac{g(z)}{1 - z} > 0.$$

But then $f'(\eta) = g'(\eta) + 1 > 1$.

Thus is both cases there is a point where $f' > 1$.

(III)(a)

(b) Plugging into f shows $x = 2,\ 4$ are fixed points. If there were 3 fixed points, say $x < y < z$, then by the Mean Value Theorem there would be $\xi \in (x, y)$ and $\eta \in (y, z)$ with $f'(\xi) = f'(\eta) = 1$. But f' is strictly increasing so this is impossible, meaning there are at most 2 fixed points, so 2 and 4 are the only ones.

(c) Let $g(x) = f(x) - x$. The g' is increasing, $g(2) = g(4) = 0,\ g'(2) < 0, g'(4) > 0$ so we must have $g(x) < 0$ (equivalently, $f(x) < x$) only for $2 < x < 4$. Also, since f is monotone,

$$x < 2 \implies f(x) < f(2) = 2,$$

$$2 < x < 4 \implies 2 = f(2) < f(x) < f(4) = 4,$$
$$x > 4 \implies 4 = f(4) < f(x).$$

In other words, applying f does not move a point outside the interval $(-\infty, 2), (2, 4)$ or $(4, \infty)$ where it started. Therefore

$$x_n < 2 \implies 2 > x_{n+1} = f(x_n) > x_n, \quad \text{for all } n,$$

meaning

$$x_0 < 2 \implies x_0 < x_1 < x_2 < ... < 2. \tag{2}$$

Also,

$$2 < x_n < 4 \implies 2 < f(x_n) < x_n < 4,$$

so

$$2 < x_0 < 4 \implies 4 > x_0 > x_1 > x_2 > ... > 2. \tag{3}$$

Finally,

$$x_n > 4 \implies x_{n+1} = f(x_n) > x_n > 4,$$

so

$$x_0 > 4 \implies 4 < x_0 < x_1 < x_2 < ... \tag{4}$$

Thus in all cases (2), (3), (4), $\{x_n\}$ is monotone.

(d) Since f is continuous, if $x_n \to x^*$ then $f(x^*) = \lim_n f(x_n) = \lim_n x_{n+1} = x^*$.

(e) For $x_0 < 2$ and $2 < x_0 < 4$, the sequence is monotone and (by (2) and (3)) bounded, hence is convergent. By (b) and (d), the limit must be 2 in both cases. Since 2 is a fixed point, the sequence is constant, hence convergent, if $x_0 = 2$, and similarly if $x_0 = 4$. For $x_0 > 4$ the sequence cannot converge to a finite limit, since by (d) and (4) this limit would have to be a fixed point greater than 4, which does not exist.

(IV) Since the second derivative in y is negative, the maximizing value $g(x)$ for y must be unique for each fixed x, so it is given as the unique solution of $(D_2 f)(x, y) = 0$. $D_2 f$ is a $\mathcal{C}'$ function from an open subset of $\mathbb{R}^2$ to $\mathbb{R}$ whose derivative has matrix $[A_x\ A_y]$ with

$A_y = D_{22}f < 0$ by assumption. By the Implicit Function Theorem (applied with the roles of x and y interchanged), g is a $\mathcal{C}'$ function.

(V)(a) Let $f = (f_1, f_2)$. In the notation of the Implicit Function Theorem, the matrices at $\mathbf{x} = (3,2), \mathbf{y} = (1,1,2)$ are

$$A_x = \begin{bmatrix} -2x_1 & 2x_2 \\ 2x_1 & 4x_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \qquad A_y = \begin{bmatrix} 2y_1 & 2y_2 & 2y_3 \\ 2y_1 & -2y_2 & 2y_3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix}.$$

Since A_x is invertible, we can locally solve for $\mathbf{x} = g(\mathbf{y})$ with

$$g'((1,1,2)) = -A_x^{-1}A_y = \frac{1}{72}\begin{bmatrix} 8 & -4 \\ -6 & -6 \end{bmatrix}\begin{bmatrix} 2 & 2 & 4 \\ 2 & -2 & 4 \end{bmatrix} = \frac{1}{72}\begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix}.$$

This means that if we move $\mathbf{y}$ in direction $\Delta\mathbf{y} = (0,1,1)$, $\mathbf{x}$ must move in direction

$$\Delta\mathbf{x} = g'((1,1,2))\Delta\mathbf{y} = \frac{1}{72}\begin{bmatrix} 8 & 24 & 16 \\ -24 & 0 & -48 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/9 \\ -2/3 \end{bmatrix}.$$

(b) We need

$$f'(x,y)\begin{bmatrix} \mathbf{h} \\ \mathbf{k} \end{bmatrix} = 0,$$

that is, $A_y\mathbf{k} = -A_x\mathbf{h}$. This can be solved for $\mathbf{k}$ if and only if $A_x\mathbf{h}$ is in the range of A_y, which is the span of the columns $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -2 \end{bmatrix}$. In other words, $\mathbf{h}$ must be in the span of $A_x^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$ and $A_x^{-1}\begin{bmatrix} 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 0 \end{bmatrix}$. But this span is two-dimensional so $\mathbf{h}$ can be arbitrary.

(c) A_x and A_y are still given by the formulas in (a) as functions of $\mathbf{x}$ and $\mathbf{y}$, which at the new values $\mathbf{x} = (3,2), \mathbf{y} = (1,0,2)$ gives

$$A_x = \begin{bmatrix} -6 & 4 \\ 6 & 8 \end{bmatrix}, \qquad A_y = \begin{bmatrix} 2 & 0 & 4 \\ 2 & 0 & 4 \end{bmatrix}.$$

The range of A_y now consists only of multiples of $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ so $\mathbf{h}$ must be a multiple of $A_x^{-1}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1/9 \\ 1/3 \end{bmatrix}$.