MATH 425b ASSIGNMENT 8 SOLUTIONS
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Chapter 10:

(2) For $y \in (2^{-i}, 2^{1-i})$ the only function $\varphi_j(y)$ which may be nonzero is $\varphi_i(y)$. Hence for such y we have $f(x,y) = [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$ for all x, and then

$$\int f(x,y)\ dx = \varphi_i(y) \int [\varphi_i(x) - \varphi_{i+1}(x)]\ dx = \varphi_i(y)(1-1) = 0.$$

Therefore $\int dy \int f(x,y)\ dx = 0$.

In the other direction, for fixed $i \geq 2$ and $x \in (2^{-i}, 2^{1-i})$ the only functions $[\varphi_j(x) - \varphi_{j+1}(x)]$ which may be nonzero are $[\varphi_i(x) - \varphi_{i+1}(x)]$ and $[\varphi_{i-1}(x) - \varphi_i(x)]$. Hence for such x we have $f(x,y) = [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y) + [\varphi_{i-1}(x) - \varphi_i(x)]\varphi_{i-1}(y)$ for all y, and then

$$\int f(x,y)\ dy = [\varphi_i(x) - \varphi_{i+1}(x)] \int \varphi_i(y)\ dy + [\varphi_{i-1}(x) - \varphi_i(x)] \int \varphi_{i-1}(y)\ dy = \varphi_{i-1}(x) - \varphi_{i+1}(x).$$

But since $x \in (2^{-i}, 2^{1-i})$ we have $\varphi_{i-1}(x) 0 \varphi_{i+1}(x) = 0$ so $\int f(x,y)\ dy = 0$. Note we have omitted $i = 1$ which corresponds to $x \in (1/2, 1)$. For the case of $x \in (1/2, 1)$, the only function $[\varphi_j(x) - \varphi_{j+1}(x)]$ which may be nonzero is $[\varphi_1(x) - \varphi_2(x)]$. Hence for such x we have $f(x,y) = [\varphi_1(x) - \varphi_2(x)]\varphi_1(y)$ for all y, and then

$$\int f(x,y)\ dy = [\varphi_1(x) - \varphi_2(x)] \int \varphi_1(y)\ dy = [\varphi_1(x) - 0] \cdot 1 = \varphi_1(x).$$

Therefore

$$\int\ dx \int f(x,y)\ dy = \int_{1/2}^{1} \varphi_1(x)\ dx = 1.$$

(3)(b) Let $f(x,y) = (y,x)$. Suppose $G_1(x,y) = (g_1(x,y), y)$ and $G_2(x,y) = (x, g_2(x,y))$ are primitive and $G_2 \circ G_1 = f$. Then $G_1(x,0) = (g_1(x,0), 0)$ so

$$(0,x) = (G_2 \circ G_1)(x,0) = G_2(g_1(x,0), 0) = (g_1(x,0), g_2(g_1(x,0), 0))$$

for all x. Thus $g_1(x,0) = 0$ for all x, so, by matching the second coordinates, we have $g_2(0,0) = x$ for all x, a contradiction. Similarly we can't have $G_1 \circ G_2 = f$.

(7)(a) $Q^k = \{x \in \mathbb{R}^k : x_i \geq 0, \sum_{i=1}^k x_i \leq 1\}$. Suppose $x, y \in Q^k$ and $\lambda \in [0,1]$. Let $z = \lambda x + (1-\lambda)y$. Then $z_i = \lambda x_i + (1-\lambda)y_i \geq 0$ for all i and

$$\sum_{i=1}^k z_i = \lambda\left(\sum_{i=1}^k x_i\right) + (1-\lambda)\left(\sum_{i=1}^k y_i\right) \leq \lambda \cdot 1 + (1-\lambda) \cdot 1 = 1,$$

so $z \in Q^k$. Thus Q^k is convex. Clearly $0, e_1, .., e_k$ are all in Q^k, so Q^k is a convex set containing $0, e_1, .., e_k$.

To show Q^k is the smallest such set, suppose C is convex and contains $0, e_1, .., e_k$. Let $x \in Q^k$. Then $x = \sum_{i=1}^k x_i e_i + (1 - x_1 - .. - x_k) \cdot 0$ which is a convex combination of $0, e_1, .., e_k$. Thus $x \in C$. It follows that $Q^k \subset C$, which means Q^k is the smallest convex set containing $0, e_1, .., e_k$.

(b) Let $f(\mathbf{x}) = \mathbf{b} + A\mathbf{x}$ be an affine map and C a convex set. Let $\mathbf{u}, \mathbf{v} \in f(C)$ and $\lambda \in [0,1]$. There exist $\mathbf{x}, \mathbf{y} \in C$ with $f(\mathbf{x}) = \mathbf{u}, f(\mathbf{y})_= \mathbf{v}$. Hence (since C is convex) $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in C$, and

$$\begin{aligned}
f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) &= \mathbf{b} + A(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \\
&= \lambda\mathbf{b} + (1-\lambda)\mathbf{b} + \lambda A\mathbf{x} + (1-\lambda)A\mathbf{y} \\
&= \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}) \\
&= \lambda\mathbf{u} + (1-\lambda)\mathbf{v},
\end{aligned}$$

so $\lambda\mathbf{u} + (1-\lambda)\mathbf{v} \in f(C)$. This shows $f(C)$ is convex.

(8) For some $\mathbf{b}$ and some matrix A we have $T(\mathbf{x}) = \mathbf{b} + a\mathbf{x}$. We have $\mathbf{b} = T((0,0)) = (1,1)$ while the columns of A are $A((1,0)) = T((1,0)) - \mathbf{b} = (3,2) - (1,1) = (2,1)$ and $A((0,1)) = T((0,1)) - \mathbf{b} = (2,4) - (1,1) = (1,3)$. Therefore $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$. Since $T'(\mathbf{x}) = A$ for all $\mathbf{x}$, we have Jacobian $J_T = \det A = 5$. We have $H = T([0,1]^2)$, and for $\mathbf{u} \in [0,1]^2$, $T_1(\mathbf{u}) = 1 + 2u_1 + u_2$, $T_2(\mathbf{u}) = 1 + u_1 + 3u_2$. Hence

$$\begin{aligned}
\int_H e^{x-y}\, dx\, dy &= \int_0^1 \int_0^1 5e^{u_1 - 2u_2}\, du_1\, du_2 \\
&= \int_0^1 5(e-1)e^{-2u_2}\, du_2 \\
&= \frac{5}{2}(e-1)(1-e^{-2}).
\end{aligned}$$

(13) Let T be as in problem 12, so $T([0,1]^k) = Q^k$ and $J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1})$. Then by Theorem 8.20,

$$\begin{aligned}
&\int_{Q^k} x_1^{r_1}\cdots x_k^{r_k}\ d\mathbf{x} \\
&\qquad = \int_{[0,1]^k} T_1(\mathbf{u})^{r_1}\cdots T_k(\mathbf{u})^{r_k}\ J_T(\mathbf{u})\ d\mathbf{u} \\
&\qquad = \int_{[0,1]^k} u_1^{r_1}(1-u_1)^{r_2+..+r_k+k-1}u_2^{r_2}(1-u_2)^{r_3+..+r_k+k-1}\cdots u_{k-1}^{r_{k-1}}(1-u_{k-1})^{r_k+1}u_k^{r_k}\ d\mathbf{u} \\
&\qquad = \left(\int_0^1 u_1^{r_1}(1-u_1)^{r_2+..+r_k+k-1}\ du_1\right)\cdots\left(\int_0^1 u_{k-1}^{r_{k-1}}(1-u_{k-1})^{r_k+1}\ du_{k-1}\right) \\
&\qquad\qquad \cdot\left(\int_0^1 u_k^{r_k}\ du_k\right) \\
&\qquad = \left(\frac{r_1!(r_2+..+r_k+k-1)!}{(r_1+..+r_k+k)!}\right)\left(\frac{r_2!(r_3+..+r_k+k-2)!}{(r_2+..+r_k+k-1)!}\right) \\
&\qquad\qquad \cdots\left(\frac{r_{k-1}!(r_k+1)!}{(r_{k-1}+r_k+2)!}\right)\left(\frac{1}{r_k+1}\right).
\end{aligned}$$

Many factorials cancel, leaving this equal to

$$\frac{r_1!r_2!\cdots r_{k-1}!(r_k+1)!}{(r_1+\cdots+r_k+k)!}\ \frac{1}{r_k+1} = \frac{r_1!r_2!\cdots r_{k-1}!r_k!}{(r_1+\cdots+r_k+k)!}.$$

(I) We have

$$\begin{aligned}
J_1(p) &= \text{ Jacobian of } (\Phi_2(p),\Phi_3(p)) = N_1(p), \\
J_2(p) &= \text{ Jacobian of } (\Phi_3(p),\Phi_1(p)) = N_2(p), \\
J_3(p) &= \text{ Jacobian of } (\Phi_1(p),\Phi_2(p)) = N_3(p),
\end{aligned}$$

and so

$$\int_\Phi \omega_f = \int_D [f_1(\Phi(p))J_1(p) + f_2(\Phi(p))J_2(p) + f_3(\Phi(p))J_3(p)]\ dp = \int_D f(\Phi(p))\cdot N(p)\ dp,$$

with the first equality being just the definition of evaluating a differential form.

(II) The wedge product is

$$\omega\wedge\omega' = (x_1x_2 + x_3^2)x_4^2\ dx_4\wedge dx_1\wedge dx_2\wedge dx_3\wedge dx_5 - 3x_4^3\ dx_4\wedge dx_2\wedge dx_1\wedge dx_3\wedge dx_5.$$

The permutation 41235 requires 3 interchanges, and the permutation 42135 requires 4, so

$$\omega\wedge\omega' = -(x_1x_2x_4^2 + x_3^2x_4^2 - 3x_4^3)\ dx_1\wedge dx_2\wedge dx_3\wedge dx_4\wedge dx_5.$$

Next, $d\omega$ has only one nonzero term:

$$d\omega = \frac{\partial}{\partial x_3}(x_1x_2 + x_3^2)\ dx_3 \wedge dx_4 \wedge dx_1 \wedge dx_2 = 2x_3\ dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4.$$

(III) Suppose $\omega = \sum_{i=1}^n f_i(x)\ dx_i$. Using the change of variable $t = \varphi(s), dt = \varphi'(s)\ ds$ and the chain rule, we get

$$\begin{aligned}\int_\gamma \omega &= \int_c^d \sum_{i=1}^n f_i(\gamma(t))\gamma_i'(t)\ dt \\ &= \int_a^b \sum_{i=1}^n f_i(\gamma(\varphi(s)))\gamma_i'(\varphi(s))\varphi'(s)\ ds \\ &= \int_a^b \sum_{i=1}^n f_i(\alpha(s))\alpha_i'(s)\ ds \\ &= \int_\alpha \omega. \qquad (1)\end{aligned}$$

(IV) One choice is $\gamma(t) = (\cos t, \sin t), t \in [0, \pi]$. This gives

$$\int_\gamma y\ dx = \int_0^\pi \sin t(-\sin t)\ dt = -\frac{\pi}{2}.$$