MATH 425b ASSIGNMENT 9 SOLUTIONS
SPRING 2007
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Chapter 10:

(16) Let $\sigma_{ij} = [p_0, .., p_{i-1}, p_i, .., p_{j-1}, p_{j+1}, .., p_k]$ (the simplex with p_i, p_j missing.) Then

$$\partial\sigma = \sum_{i=0}^{k} (-1)^i [p_0, .., p_{i-1}, p_{i+1}, .., p_k],$$

$$\partial^2\sigma = \sum_{i=0}^{k} (-1)^i \left(\sum_{j<i} (-1)^j \sigma_{ji} + \sum_{j>i} (-1)^{j-1} \sigma_{ij} \right)$$

A given σ_{ij} $(= \sigma_{ji})$, say σ_{24}, appears once with $i = 2, j = 4$ and once with $j = 2, i = 4$, giving a contribution of

$$(-1)^2\sigma_{24} + (-1)^3\sigma_{24} = 0.$$

Similarly we get, after switching the names of the indices in the second sum,

$$\begin{aligned} \partial^2\sigma &= \sum_{i=0}^{k} (-1)^i \sum_{j<i} (-1)^j \sigma_{ji} + \sum_{j=0}^{k} (-1)^j \sum_{i>j} (-1)^{i-1} \sigma_{ji} \\ &= \sum_{(i,j):j<i} (-1)^{i+j} \sigma_{ji} + \sum_{(i,j):j<i} (-1)^{i+j+1} \sigma_{ji} \\ &= 0 \end{aligned}$$

For a chain $\Psi = \Phi_1 + .. + \Phi_r$, we have $\partial^2\Psi = \sum_{i=1}^{r} \partial^2\Phi_i$ so it's enough to show $\partial^2\Phi = 0$ for all surfaces $\Phi = T \circ \sigma$ (where σ is affine and T is $\mathcal{C}''$, as in 10.30.) Then $\partial\Phi = T(\partial\sigma)$ and $\partial^2\Phi = T(\partial^2\sigma) = 0$ since $\partial^2\sigma = 0$.

(20) Suppose Φ is a k-surface of class $\mathcal{C}''$ in an open $V \subset \mathbb{R}^n$, ω is a $(k-1)$-form of class $\mathcal{C}'$ in V, and f is a $\mathcal{C}'$ function on V. Then $f\omega$ is a $(k-1)$-form of class $\mathcal{C}'$ on V so by Stokes Theorem and 10.20a,

$$\int_{\partial\Phi} f\omega = \int_{\Phi} d(f\omega) = \int_{\Phi} df \wedge \omega + \int_{\Phi} f \, d\omega.$$

(21)(a) For $\eta = \frac{x\ dy - y\ dx}{x^2+y^2}$ we have

$$
\begin{aligned}
d\eta &= \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right)\ dx \wedge dy - \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\ dy \wedge dx \\
&= \frac{x^+y^2 - 2x^2}{(x^2+y^2)^2}\ dx \wedge\ dy + \frac{x^+y^2 - 2y^2}{(x^2+y^2)^2}\ dx \wedge\ dy \\
&= 0.
\end{aligned}
$$

(b) For $\Phi(t,u) = (1-u)\Gamma(t) + u\gamma(t)$ $(0 \le t \le 2\pi, 0 \le u \le 1)$ we have $\partial\Phi = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$ where

$$\gamma_1 : [0, 2\pi] \to E, \quad \gamma_1(t) = \Phi(t,0) = \Gamma(t);$$

$$\gamma_2 : [0,1] \to E, \quad \gamma_2(u) = \Phi(2\pi, u) = (1-u)\Gamma(0) + u\gamma(0)$$

(which traces a line from $\Gamma(0)$ to $\gamma(0)$);

$$\gamma_3 : [0, 2\pi] \to E, \quad \gamma_3(t) + \Phi(2\pi - t, 1) = \gamma(2\pi - t);$$

$$\gamma_4 : [0,1] \to E, \quad \gamma_4(u) = \Phi(0, 1-u) = u\Gamma(0) + (1-u)\gamma(0).$$

Each γ_i is the image of one side of the rectangle, going counterclockwise. Also $\gamma_4 = -\gamma_2$ and $\gamma_3 = -\gamma$ (as surfaces), so

$$
\begin{aligned}
0 &= \int_\Phi d\eta \\
&= \int_{\partial\Phi} \eta \\
&= \int_{\gamma_1} \eta + \int_{\gamma_2} \eta + \int_{\gamma_3} \eta + \int_{\gamma_4} \eta \\
&= \int_{\gamma_1} \eta + \int_{\gamma_3} \eta \\
&= \int_\Gamma \eta - \int_\gamma \eta.
\end{aligned}
$$

(d) In any open set where $x \ne 0$,

$$\frac{\partial}{\partial x} \arctan\frac{y}{x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2} = \frac{-y}{x^2+y^2},$$

$$\frac{\partial}{\partial y} \arctan\frac{y}{x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2+y^2},$$

so

$$d\left(\arctan\frac{y}{x}\right) = \frac{x\ dy - y\ dx}{x^2+y^2} = \eta.$$

Similarly, where $y \neq 0$,

$$d\left(-\arctan\frac{x}{y}\right) = -\frac{y\ dx - x\ dy}{x^2+y^2} = \eta.$$

(24) Let $\omega = \sum_i a_i(\mathbf{x})\ dx_i$ be a 1-form of class $\mathcal{C}''$ in E with $d\omega = 0$, and let $\mathbf{p} \in E$. Define $f(\mathbf{x}) = \int_{[\mathbf{p},\mathbf{x}]} \omega$, for $x \in E$. By Stokes Theorem, for $\mathbf{x} \neq \mathbf{y}$ and $\gamma(t) = (1-t)\mathbf{x} + t\mathbf{y}$,

$$0 = \int_{[\mathbf{p},\mathbf{x},\mathbf{y}]} d\omega = \int_{\partial[\mathbf{p},\mathbf{x},\mathbf{y}]} \omega = \int_{[\mathbf{p},\mathbf{x}]} \omega - \int_{[\mathbf{p},\mathbf{y}]} \omega + \int_{[\mathbf{x},\mathbf{y}]} \omega$$

$$= f(\mathbf{x}) - f(\mathbf{y}) + \int_0^1 \sum_{i=1}^n a_i(\gamma(t))\gamma_i'(t)\ dt = f(\mathbf{x}) - f(\mathbf{y}) + \sum_{i=1}^n (y_i - x_i) \int_0^1 a_i(\gamma(t))\ dt.$$

Taking $\mathbf{y} = \mathbf{x} + he_i$ we get

$$\frac{f(\mathbf{y}) - f(\mathbf{x})}{h} = \int_0^1 a_i(\mathbf{x} + the_i)\ dt \to a_i(\mathbf{x}) \quad \text{as } h \to 0,$$

since $the_i \to 0$ uniformly in t, as $h \to 0$. Thus $(D_i f)(\mathbf{x}) = a_i(\mathbf{x})$, so $\omega = df$.

(25) It is enough to prove exactness separately in each connected component of E, so we may assume E is connected. Fix $\mathbf{p}$ and $\mathbf{x}$ in E and let γ_1, γ_2 be any two $\mathcal{C}'$ paths from $\mathbf{p}$ to $\mathbf{x}$. We assume these are parametrized by $[0,1]$ in such a way that $\gamma_i'(1) = 0$. This ensures that the path γ obtained by following γ_1 for $\mathbf{p}$ to $\mathbf{x}$ followed by γ_2 backwards from $\mathbf{x}$ to $\mathbf{p}$ is a $\mathcal{C}'$ closed curve, and hence by assumption

$$0 = \int_\gamma \omega = \int_{\gamma_1} \omega - \int_{\gamma_2} \omega. \tag{1}$$

It follows that we can define $f(\mathbf{x}) = \int_{\Gamma_\mathbf{x}} \omega$ where $\Gamma_\mathbf{x}$ is any $\mathcal{C}'$ path from $\mathbf{p}$ to $\mathbf{x}$, since this will not depend on the choice of $\Gamma_\mathbf{x}$. A fixed $\mathbf{x} \in E$ has a convex open neighborhood in E, and for $\mathbf{y} \in E$ we can take $\Gamma_\mathbf{y}$ to be a path from $\mathbf{p}$ to $\mathbf{x}$ followed by a straight line from $\mathbf{x}$ to $\mathbf{y}$. The proof in Exercise 24 then applies to show that $\omega = df$.

Note that in the case of $E = \mathbb{R}^2\backslash\{0\}$ we can take $\Gamma_\mathbf{x}$ to be any path not passing through 0, and there is always such a path consisting of at most two line segments. This means that we really only need (1) for closed paths γ consisting of at most 4 line segments, as we simply require $\Gamma_\mathbf{x}$ to be a path of at most two line segments, in the definition of $f(\mathbf{x})$.

(I) Let $\omega = M\ dx + N\ dy$, so $d\omega = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx \wedge\ dy$. Therefore the following are equivalent:

(i) the differential equation is exact;

(ii) $d\omega = 0$;

(iii) ω is closed;

(iv) ω is exact;

(v) $\omega = dF$ for some F;

(vi) $\omega = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$;

(vii) $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

(II) Define $\Phi : [0,1]^2 \to \mathbb{R}^3$ by $\Phi(x,y) = (x, y, x^4 + y^2)$. Let

$$J_1(x,y) = \text{ Jacobian of } (y, x^4 + y^2) = \det \begin{bmatrix} 0 & 1 \\ 4x^3 & 2y \end{bmatrix} = -4x^3,$$

$$J_2(x,y) = \text{ Jacobian of } (x^4 + y^2, x) = \det \begin{bmatrix} 4x^3 & 2y \\ 1 & 0 \end{bmatrix} = -2y,$$

$$J_3(x,y) = \text{ Jacobian of } (x,y) = \det I = 1.$$

Then

$$\begin{aligned}
\int_\Phi \omega &= \int_0^1 \int_0^1 \left[xJ_1(x,y) + yJ_2(x,y) + (x^4+y^2)J_3(x,y) \right] \ dx \ dy \\
&= \int_0^1 \int_0^1 \left[-4x^4 - 2y^2 + x^4 + y^2 \right] \ dx \ dy \\
&= \int_0^1 \left[\left(-3\frac{x^5}{5} - y^2 x \right) \Big|_0^1 \right] \ dy \\
&= \int_0^1 \left(-\frac{3}{5} - y^2 \right) \ dy \\
&= \left(-\frac{3}{5}y - \frac{1}{3}y^3 \right) \Big|_0^1 \\
&= -\frac{3}{5} - \frac{1}{3} \\
&= -\frac{14}{15}.
\end{aligned}$$

(III) Use Stokes Theorem: $\int_S \omega = \int_E d\omega$. We have $d\omega = (yz + 2y + 1)\ dx \wedge dy \wedge dz$ so

$$\int_E d\omega = \int_{-2}^{2} \int_{x^2}^{4} \int_{0}^{1} (yz + 2y + 1)\ dz\ dy\ dx.$$

Now

$$\int_0^1 (yz + 2y + 1)\ dz = (y\frac{z^2}{2} + 2yz + z)\Big|_0^1 = \frac{5}{2}y + 1,$$

while

$$\int_{x^2}^{4} (\frac{5}{2}y + 1)\ dy = (\frac{5}{4}y^2 + y)\Big|_{x^2}^{4} = 24 - \frac{5}{4}x^4 - x^2,$$

so

$$\int_E d\omega = \int_{-2}^{2} (24 - \frac{5}{4}x^4 - x^2)\ dx = (24x - \frac{1}{4}x^5 - \frac{1}{3}x^3)\Big|_{-2}^{2} = \frac{224}{3}.$$