MATH 425b SAMPLE FINAL EXAM
Spring 2009
Prof. Alexander

This final was an "extra short" one given along with a take-home final. Since we have only the in-class final, ours won't be quite so short!

(1)(25 points)(a) Let $\omega = (x^2y^2 + z^2)\, dx \wedge dy + z^3\, dx \wedge dz$ in $\mathbb{R}^3$. Show that for every cube $A = [-a, a]^3$ centered at the origin, we have

$$\int_{\partial A} \omega = 0. \tag{1}$$

HINT: It is not necessary to parametrize ∂A to do this.

(b) For what other points (x_0, y_0, z_0) does (1) remain true for all cubes A centered at (x_0, y_0, z_0)?

(2)(25 points) Recall the norm on $n \times n$ matrices given by

$$||A|| = \sup_{x:|x|\leq 1} |Ax| = \sup_{x\neq 0} \frac{|Ax|}{|x|}.$$

(a) Show that

$$\sup_{x,y\neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \leq ||A||.$$

Here $Ax \cdot y$ is the dot product.

(b) Prove the reverse inequality:

$$\sup_{x,y\neq 0} \frac{|Ax \cdot y|}{|x|\ |y|} \geq ||A||.$$

HINT for (b): Compare the sup to a particular choice of y.

(3)(25 points)(a) Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\mathbf{x}$ with $f(\mathbf{x}) \neq 0$. Show directly from the definition of derivative that $1/f$ is differentiable at $\mathbf{x}$, and express the derivative of $1/f$ in terms of $f'(\mathbf{x})$ and $f(\mathbf{x})$.

HINT: "Directly from the definition" means don't use the Chain Rule or other theorems. Use

$$\frac{1}{f(\mathbf{x}+\mathbf{h})} - \frac{1}{f(\mathbf{x})} + \frac{f'(\mathbf{x})\mathbf{h}}{f(\mathbf{x})^2} = \frac{f(\mathbf{x}) - f(\mathbf{x}+\mathbf{h}) + f'(\mathbf{x})\mathbf{h}}{f(\mathbf{x})f(\mathbf{x}+\mathbf{h})} + (\text{what??}).$$

(b) Let $g : \mathbb{R}^n \to \mathbb{R}^m$ and suppose there is a neighborhood U of 0 in $\mathbb{R}^n$ and an $\alpha > 1$ such that $|g(\mathbf{x})| \leq |\mathbf{x}|^\alpha$ for all $\mathbf{x} \in U$. Show that g is differentiable at 0, and find $g'(0)$. HINT: To help you guess $g'(0)$ before you prove it, consider the function $g(x) = |x|^\alpha$ on $\mathbb{R}$.