MATH 425b SAMPLE MIDTERM EXAM 1
Spring 2009
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The midterm will be open book. You can use Rudin, your lecture notes, your homework and solutions, but no other books or published materials.

(1) Let $\alpha > 0$ and define

$$E = \{f \in C[0,1] : f(0) = 0,\ |f(y) - f(x)| \leq |y - x|^\alpha \text{ for all } x, y\}.$$

(a) Show that E is equicontinuous.
(b) Show that E is a closed subset of $C[0,1]$ (with the uniform metric, as always.)

(2) You may take the following as given: the function $f(x) = (1-x)^{-1/2}$ has Taylor series $\sum_{n=0}^\infty \frac{f^{(n)}(0)}{n!} x^n$ and the coefficients $a_n = \frac{f^{(n)}(0)}{n!}$ satisfy $a_n > 0, a_n \searrow 0$. For every x for which the series converges, the sum is $f(x)$.

(a) Show that the radius of convergence is at least 1, that is, the series converges for $|x| < 1$.

(b) *Without calculating the coefficients* a_n, show that they must satisfy $\sum_n a_n = \infty$. HINT: Consider $x \nearrow 1$. Proceed by contradiction.

(3) Recall that the nth Fourier coefficient of a function f is defined to be $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$.

(a) Let $f(x) = \sum_{n=-N}^{N} a_n e^{inx}$ be a trigonometric polynomial. Show that a_n is the nth Fourier coefficient of f.

(b) Suppose $a_n \in \mathbb{C}$ and the series $f(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges uniformly on $[-\pi, \pi]$. (We don't assume this is the Fourier series of f, just that it's some series that converges uniformly to f.) Show that a_n must be the nth Fourier coefficient of f. HINT: Be careful about interchanging sums and integrals.

(4) For this problem, $||f||_\infty$ means $\sup_{x\in\mathbb{R}} |f(x)|$, i.e. the sup is over all of $\mathbb{R}$.

(a) Show that if P is a nonconstant polynomial then $||P||_\infty = \infty$. HINT: If P has degree k, consider $P(x)/x^k$ as $x \to \infty$.

(b) Show that if a function $f : \mathbb{R} \to \mathbb{R}$ is a uniform limit of polynomials (that is, $||P_n - f||_\infty \to 0$), then f itself is a polynomial. HINT: Consider $P_n - P_m$ in (a). Note if f is a polynomial then taking $P_n = f$ for all n is allowed.