MATH 525b IN-CLASS FINAL EXAM
May 8, 2009
Prof. Alexander

Last Name: ______________________________

First Name: ______________________________

USC ID: ______________________________

Signature: ______________________________

Problem	Points	Score
1	15	
2	20	
3	25	
4	20	
Total	80	

Notes:

(1) This is an open-book exam. You may use Folland, your lecture notes, homework and solutions, and any notes you write yourself, but not other published or photocopied materials.
(2) Write on the backs of the sheets if you need more space.
(3) Cross out anything you don't want counted when the exam is graded.
(4) Problem parts with a * are harder or longer—do them only if you have time.

(1)(15 points) Let X be a normed vector space and suppose X^* is finite-dimensional. Show that X must be finite-dimensional.

(2)(20 points) Let H be a Hilbert space. A map $B : H \times H \to \mathbb{C}$ is called a *sesquilinear form* if it is linear in the first variable and conjugate-linear in the second:

$$B(ax + by, z) = aB(x, z) + bB(y, z), \qquad B(x, ay + bz) = \bar{a}B(x, y) + \bar{b}B(x, z).$$

A sesquilinear form is *bounded* if there exists M such that $B(x, y) \leq M\|x\|\|y\|$ for all $x, y \in H$.

(a) Recall Theorem 5.25: every $f \in H^*$ has the form $f(x) = \langle x, y\rangle$ for some fixed y. Show that $\|f\| = \|y\|$.

(b*) Show that for every bounded sesquilinear form B, there is a bounded operator $T : H \to H$ such that $B(x, y) = \langle Tx, y\rangle$. HINT: For fixed x consider the map $A_x : H \to H$ given by $A_x(y) = \overline{B(x, y)}$. First look at properties of $A_x(y)$ as a function of y, then look at properties of A_x as a function of x.

(3)(25 points) Suppose T_1 is a normal operator on a Hilbert space H_1, H_2 is another Hilbert space, and the operator T_2 on H_2 is unitarily equivalent to T_1, say $T_2 = VT_1V^{-1}$ for some unitary V from H_1 onto H_2.

(a) Show that T_2 is normal.

(b) Show that T_1 and T_2 have the same spectrum (call it K.)

(c) After picking an arbitrary $x_1 \in H_1$ and $x_2 \in H_2$, one obtains the measures μ_1 and μ_2 on spaces Y_1 and Y_2, respectively, which define the L^2 spaces that appear in the Spectral Theorem. Given x_1, how can you choose x_2 so that $\mu_1 = \mu_2$? (Prove.) HINT: Where does μ_i come from, in the proof of the Spectral Theorem?

(d*) For simplicity, assume x_i is chosen so that the closed linear span of $\{T_i^n(T_i^*)^m x_i : n, m \geq 0\}$ is all of H_i for $i = 1, 2$, so that when Y_i is expressed as a disjoint union of layers, there is only one layer, equal to $\sigma(T_i) = K$. In other words, assume $Y_1 = Y_2 = K$. Let E_1 and E_2 be the resolutions of the identity on K corresponding to T_1 and T_2 respectively. Assume $\mu_1 = \mu_2$ as in (c). How are E_1 and E_2 related? In other words, express $E_2(A)$ for a general set A in terms of $E_1(A)$, using perhaps V, T_1 or T_2.

(4)(20 points) The function $1/x$ on $\mathbb{R}$ is not in L^1_{loc}, but nonetheless we define the "principal value"

$$T(\varphi) = \lim_{\epsilon \to 0} \left[\int_{-\infty}^{-\epsilon} \frac{1}{x}\varphi(x)\ dx + \int_{\epsilon}^{\infty} \frac{1}{x}\varphi(x)\ dx \right], \qquad \varphi \in \mathcal{D}(\mathbb{R}).$$

(a) Show that this limit always exists and is finite. Express the value without involving any limit. HINT: Change variable to combine the two integrals into one.

(b) Show that $T \in \mathcal{D}'(\mathbb{R})$.

(c*) Find an explicit formula for $(\partial T)(\varphi)$ as an integral involving φ (not its derivatives), without a limit involved. HINT: Use the function $\psi(x) = \varphi(x) - \varphi(0) - x\varphi'(0)$.