MATH 525b IN-CLASS FINAL EXAM SOLUTIONS
SPRING 2009
Prof. Alexander

(1) FIRST PROOF: Since X^* is finite dimensional, it is isomorphic to its dual X^{**}, so X^{**} is finite-dimensional. The map $x \mapsto \hat{x}$ is an isometry from X into X^{**} so X is isometric to a subspace of the finite dimensional space X^{**}. Therefore X is finite-dimensional.

SECOND PROOF: Suppose X is infinite dimensional. Then there exist linearly independent vectors $x_1, x_2, \ldots$ in X. Fix $n \geq 1$. By Theorem 5.8 there exist $f_1, \ldots, f_n \in X^*$ with $f_i(x_j) = 0$ for $i, j \leq n$ with $j \neq i$, and $f_i(x_i) = 1$ for $i \leq n$. We claim these f_i are linearly independent. In fact, if $\sum_{i=1}^n c_i f_i = 0$ then for each j we have $0 = \sum_{i=1}^n c_i f_i(x_j) = c_j$. Thus the dimension of X^* is at least n, for all n, so this dimension is infinite.

(2)(a) We have $|f(x)| = |\langle x, y\rangle| \leq \|x\|\|y\|$ for all x, so $\|f\| \leq \|y\|$. In the other direction, if $f \neq 0$ then $y \neq 0$ so $\underline{\|f\| \geq} f(y/\|y\|) = \langle y, y\rangle/\|y\| = \|y\|$. Thus $\|f\| = \|y\|$.

(b) Let $A_x(y) = \overline{B(x, y)}$. From the definition of sesquilinear, A_x is linear. Also $|A_x y| = |B(x, y)| \leq M\|x\|\|y\|$ for all y so $\|A_x\| \leq M\|x\| < \infty$. Therefore A_x is a bounded linear functional, so there exists w_x such that $A_x y = \langle y, w_x\rangle$ for all y and $\|w_x\| = \|A_x\| \leq M\|x\|$. For $x, v \in H$ and c a scalar we have for all y that

$$\langle y, w_{cx+v}\rangle = A_{cx+v}y = \overline{B(cx + v, y)} = \bar{c}\overline{B(x, y)} + \overline{B(v, y)} = \bar{c}\langle y, w_x\rangle + \langle y, w_v\rangle = \langle y, cw_x + w_v\rangle,$$

so $w_{cx+v} = cw_x + w_v$, meaning that the map $Tx = w_x$ is linear. As noted we have $\|w_x\| \leq M\|x\|$ so T is bounded. Therefore we have

$$B(x, y) = \overline{A_x(y)} = \langle w_x, y\rangle = \langle Tx, y\rangle.$$

(3)(a) $T_2^* = (V^{-1})^* T_1^* V^* = V T_1^* V^{-1}$ so $T_2^* T_2 = V T_1^* V^{-1} \cdot V T_1 V^{-1} = V T_1^* T_1 V^{-1} = V T_1 T_1^* V^{-1} = V T_1 V^{-1} \cdot V T_1^* V^{-1} = T_2 T_2^*$. Thus T_2 is normal.

(b) We have $\lambda \notin \mathrm{Spec}(T_i)$ if and only if $T_i - \lambda I$ has an inverse S defined on all of H_i. (Note having the inverse defined on all of H_i means the range of $T_i - \lambda I$ is all of H_i.) Now

$$\begin{aligned} T_2 - \lambda I \text{ has inverse } S &\iff (T_2 - \lambda I)S = S(T_2 - \lambda I) = I \\ &\iff V^{-1}(T_2 - \lambda I)V \cdot V^{-1}SV = V^{-1}SV \cdot V^{-1}(T_2 - \lambda I)V = I \\ &\iff V^{-1}(T_2 - \lambda I)V \text{ has inverse } V^{-1}SV \\ &\iff T_1 - \lambda I \text{ has inverse } V^{-1}SV. \end{aligned}$$

Thus $T_2 - \lambda I$ is invertible if and only if $T_1 - \lambda I$ is invertible, so they have the same spectrum.

(c) For $p(z, \bar{z})$ a polynomial, let $S_i(p) = \langle p(T_i, T_i^*)x_i, x_i\rangle$. From the proof of the Spectral Theorem, S_i determines μ_i, so it is enough to show $S_1 = S_2$. We have $p(T_2, T_2^*) =$

$Vp(T_1, T_1^*)V^{-1}$ so

$$S_1(p) = \langle p(t_1, T_1^*)x_1, x_1\rangle = \langle V^{-1}p(T_2, T_2^*)Vx_1, x_1\rangle = \langle p(t_2, T_2^*)Vx_1, Vx_1\rangle$$

By taking $x_2 = Vx_1$ we thus get $S_1 = S_2$ so $\mu_1 = \mu_2$ (call the common value μ.)

(d) We claim that $E_2(A) = VE_1(A)V^{-1}$ for all A. For $i = 1, 2$ there is an isometry U_i from $L^2(K, \mu)$ onto H_i such that for ϕ the identity on K, $E_i(A) = U_i M_{\chi_A \circ \phi} U_i^{-1}$. For polynomials $p(z, \bar{z})$, U_i is given by $U_i(p) = p(T_i, T_i^*)x_i$, so

$$U_2(p) = p(T_2, T_2^*)x_2 = Vp(T_1, T_1^*)V^{-1}Vx_1 = Vp(T_1, T_1^*)x_1 = VU_1(p),$$

and since such polynomials are dense in $L^2(K, \mu)$, we have $U_2 = VU_1$. Therefore $V = U_2U_1^{-1}$ and

$$E_1(A) = U_1 M_{\chi_A \circ \phi} U_1^{-1} = U_1 U_2^{-1} E_2(A) U_2 U_1^{-1} = V^{-1} E_2(A) V.$$

(4)(a) Changing variable, x to $-x$, in the first integral gives

$$T(\varphi) = \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{1}{x} \left[\varphi(x) - \varphi(-x)\right] \, dx.$$

Now $|\frac{1}{x}\left[\varphi(x) - \varphi(-x)\right]| \leq 2\|\varphi'\|_\infty$, so the integrand is bounded on $(0, \infty)$, and it is 0 outside some compact interval. Therefore the limit exists and

$$T(\varphi) = \int_{(0,\infty)} \frac{1}{x} \left[\varphi(x) - \varphi(-x)\right] \, dx.$$

(b) Clearly T is linear, so we show it's continuous. Suppose $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R})$. Then there exist N and A such that $\mathrm{supp}(\varphi_n) \subset [-A, A]$ for all $n \geq N$, and φ and all derivatives converge uniformly on $[-A, A]$. Therefore there exists $M < \infty$ such that $\|\varphi_n'\|_\infty \leq M$ for all $n \geq N$. Therefore

$$\left|\frac{1}{x}\left[\varphi_n(x) - \varphi_n(-x)\right]\right| \leq 2\|\varphi_n'\|_\infty \leq M \quad \text{for all } n \geq N, x \in (0, A].$$

Therefore by bounded convergence we have

$$T(\varphi_n) = \int_{(0,A]} \frac{1}{x} \left[\varphi_n(x) - \varphi_n(-x)\right] \, dx \to T(\varphi).$$

(c) Choose A so $\mathrm{supp}(\varphi) \subset [-A, A]$. Let $\psi(x) = \varphi(x) - x\varphi'(0) - \varphi(0)$. Then

$$\psi'(x) - \psi'(-x) = \varphi'(x) - \varphi'(-x), \quad \frac{\psi(x) + \psi(-x)}{x^2} \to \varphi''(0) \text{ as } x \to 0,$$

$$\psi(x) + \psi(-x) = -2\varphi(0) \text{ for } |x| \geq A.$$

Therefore all integrals are well-defined in the following computation:

$$\begin{aligned}
(\partial T)(\varphi) &= -T(\varphi') \\
&= -\int_0^\infty \frac{1}{x} \left[\varphi'(x) - \varphi'(-x)\right] \, dx \\
&= -\int_0^\infty \frac{1}{x} \left[\psi'(x) - \psi'(-x)\right] \, dx \\
&= \frac{1}{x} \left[\psi(x) + \psi(-x)\right] \Big|_0^\infty - \int_0^\infty \frac{1}{x^2} \left[\psi(x) + \psi(-x)\right] \, dx \\
&= -\int_0^\infty \frac{1}{x^2} \left[\psi(x) + \psi(-x)\right] \, dx \\
&= -\int_0^\infty \frac{\varphi(x) + \varphi(-x) - 2\varphi(0)}{x^2} \, dx.
\end{aligned}$$