MATH 525b—CLARIFICATION ON SPECTRAL THEORY
SPRING 2009
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Recall the Spectral Theorem for Normal Operators:

Theorem. Let T be a normal operator on a separable Hilbert space X. There exists a resolution of the identity F on $(\sigma(T), \mathcal{M})$ such that

$$T = \int_{\sigma(T)} \lambda \; F(d\lambda).$$

In my presentation of this result I was not sufficiently careful about distringuishing between $\sigma(T) \subset \mathbb{C}$ and the "disjoint union of layers" $Y = \cup_{k \geq 1}(\{k\} \times Y_k)$. We know that T is unitarily equivalent to the multiplication operator M_ϕ on $L^2(Y, \mu)$, with $\phi(\lambda) = \lambda$, that is, $T = UM_\phi U^{-1}$. The key point is that this function ϕ must properly be viewed not as the identity on Y but as a map from Y to $\mathbb{C}$, taking values in $\sigma(T)$. (You can think of ϕ as "projecting" all the layers of Y back into the one layer $\sigma(T)$, and the formula $\phi(\lambda) = \lambda$ as a sort of shorthand for this.) Thus for example, for a set $A \subset \sigma(T)$, the indicator χ_A does not make sense as a function on Y, but $\chi_A \circ \phi$ does.

A modified outline of what we did in lecture, adjusted for this change, is as follows. For $A \subset \sigma(T)$ let

$$F(A) = M_{\chi_A \circ \phi},$$

which is a multiplication operator on $L^2(Y, \mu)$. For $f, g \in L^2(Y, \mu)$,

$$F_{fg}(A) = \langle F(A)f, g\rangle = \int_Y (\chi_A \circ \phi) f\bar{g} \; d\mu = \int_{\phi^{-1}(A)} f\bar{g} \; d\mu \tag{1}$$

defines a complex measure on the Borel sets in $\sigma(T)$. Then for arbitrary Borel $A \subset \sigma(T)$ and bounded measurable ψ on $\sigma(T)$,

$$\begin{aligned} \int_A \psi(\lambda) \; \langle F(d\lambda)f, g\rangle &= \int_A \psi(\lambda) \; F_{fg}(d\lambda) \\ &= \int_{\phi^{-1}(A)} (\psi \circ \phi) f\bar{g} \; d\mu \\ &= \int_Y (\chi_A \circ \phi)(\psi \circ \phi) f\bar{g} \; d\mu \qquad (2) \\ &= \langle M_{(\chi_A \psi)\circ\phi} f, g\rangle. \end{aligned}$$

Here the second equality comes from (1) when ψ is an indicator, and extends to general ψ in the standard way.

Next, for $A \subset \sigma(T)$, define the operator $E(A) = UM_{\chi_A \circ \phi} U^{-1}$. Then for $y, z \in X$,

$$\langle E(A)y, z \rangle = \langle F(A)U^{-1}y, U^{-1}z \rangle,$$

which says that E_{yz} and $F_{U^{-1}y, U^{-1}z}$ are the same measure. This means that for arbitrary Borel $A \subset \sigma(T)$ and bounded measurable ψ on $\sigma(T)$, and $y, z \in X$, using (2),

$$\begin{aligned}\int_A \psi(\lambda)\ \langle E(d\lambda)y, z \rangle &= \int_A \psi(\lambda)\ \langle F(d\lambda)U^{-1}y, U^{-1}z \rangle \\ &= \langle M_{(\chi_A \psi)\circ\phi} U^{-1}y, U^{-1}z \rangle \\ &= \langle UM_{(\chi_A \psi)\circ\phi} U^{-1}y, z \rangle.\end{aligned} \tag{3}$$

The fact this equality holds for all y, z means that $UM_{(\chi_A \psi)\circ\phi} U^{-1} = \int_A \psi(\lambda)\ E(d\lambda)$. In particular if we take ψ to be the identity and $A = \sigma(T)$, we get

$$T = UM_\phi U^{-1} = \int_{\sigma(T)} \lambda\ E(d\lambda),$$

which is the statement of the above Theorem.