MATH 525a TAKE HOME FINAL EXAM
Spring 2009
Prof. Alexander

80 points maximum.

What is allowed: Use of Folland and other published works (including internet, but not to communicate with others), your lecture notes, homework and midterms, and homework and midterm solutions.

What is not allowed: Use of nonpublished materials (e.g. someone else's lecture notes or solutions); *consulting with each other or with anyone except me.* The latter is grounds for a 0 on the exam!

Turning in the exam: The exam is due Friday May 8 at the 8 a.m. start of the in-class final.

(1)(26 points)(a) Let μ be a signed Borel measure on $\mathbb{R}^n$ which is finite on compact sets. Show that $T_\mu(\varphi) = \int \varphi \, d\mu$ defines a distribution in $\mathcal{D}'(\mathbb{R}^n)$.

(b) Suppose S and T are distributions in $\mathcal{D}'(\mathbb{R})$. Show that the distributional derivatives satisfy $\partial S = \partial T$ if and only if there is a constant c such that

$$S(\varphi) = T(\varphi) + c \int \varphi \, dm \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}).$$

(c) Give a characterization of those functions $f \in L^1_{loc}(\mathbb{R})$ for which the distributional derivative $\partial f = T_\mu$ for some μ as in (a). HINT: See Theorem 2 and its proof. Also, which functions in $\mathcal{D}(\mathbb{R})$ are the (classical) derivative of some other function in $\mathcal{D}(\mathbb{R})$?

(2)(22 points) Folland Ch. 7 #27, p. 225.

(3)(32 points) In the notation used in lecture, let H be a Hilbert space, let $U : L^2(Y, \mu) \to H_x$ be the isometry associated to a normal operator $T : H \to H$ and a fixed $x \in H$, and let $E(\cdot)$ be the resolution of the identity given by $E(A) = UM_{\chi_A}U^{-1}$.

(a) Show that for the complex measures E_{yz}, on the Borel sets $\mathcal{B}_Y$, given by $E_{yz}(A) = \langle E(A)y, z\rangle$ for $y, z \in H$, we have

$$E_{yy}(A) \geq 0 \text{ for all } y, A, \quad \text{and} \quad \sup_{||y||=1, ||z||=1} |E_{yz}|(Y) < \infty.$$

Here $|E_{yz}|$ denotes the total variation.

(b) Show that if $\psi_k \to \psi$ in $L^\infty(Y, \mu)$ (with essential-sup norm), then the operators $\int \psi_k(\lambda) \, E(d\lambda) \to \int \psi(\lambda) \, E(d\lambda)$ in norm.

(c) Show that if $S : H \to H$ is another operator which commutes with $E(A)$ for all measurable $A \subset Y$, then S commutes with $\int \psi(\lambda)\; E(d\lambda)$ for all $\psi \in L^\infty(Y, \mu)$.

(d) Let $\lambda_0 \in \sigma(T)$, let B_r denote the ball of radius r centered at λ_0 and let W_r denote the range of the projection $E(B_r)$. Show that as $r \to 0$, W_r becomes more and more like an eigenspace, in the sense that $\|(T - \lambda_0 I)|_{W_r}\| \to 0$. Here $(T - \lambda_0 I)|_{W_r}$ denotes the restriction of $T - \lambda_0 I$ to W_r. HINT: Find a bound for $\left|\langle (T - \lambda_0 I)E(B_r)y, z\rangle\right|$. It may be useful to use the unitary map U.