MATH 525b ASSIGNMENT 2 SOLUTIONS
SPRING 2009
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Chapter 4

(5) Let $\mathcal{X}$ be a metric space with a countable dense subset E. Let $\mathcal{B}$ be the set of all balls with center in E and rational radius. Then $\mathcal{B}$ is countable, and we claim $\mathcal{B}$ is a base. Let $x \in \mathcal{X}$ and let U be an open set containing x. Then the open ball $B(r,x) \subset U$ for some $r > 0$. Since E is dense, there is a $y \in E$ with $d(x,y) < r/2$. Let q be a rational with $d(x,y) < q < r/2$. We claim that $B(q,y) \subset B(r,x)$. In fact if $d(z,y) < q$ then $d(z,x) \leq d(z,y) + d(y,x) < q + r/2 < r$, proving this claim. Thus $B(q,y) \in \mathcal{B}$ and $x \in B(q,y) \subset B(r,x) \subset U$. Thus $\mathcal{B}$ is a base.

(16)(a) Let $F = \{x : f(x) = g(x)\}$ and $y \notin F$. Then $f(y) \neq g(y)$, so since Y is Hausdorff, there exist disjoint neighborhoods U, V of $f(y), g(y)$ respectively. Since f, g are continuous, $f^{-1}(U)$ and $g^{-1}(V)$ are both open neighborhoods of y, hence so is $f^{-1}(U) \cap g^{-1}(V)$. But since U, V are disjoint, $x \in f^{-1}(U) \cap g^{-1}(V)$ implies $f(x) \neq g(x)$, that is, $x \notin F$. Thus y has a neighborhood $f^{-1}(U) \cap g^{-1}(V)$ contained in F^c, so F^c is open. Equivalently, F is closed.

(17) Suppose that

$$\text{for all } x \neq y \text{ there exists } f \in \mathcal{F} : f(x) \neq f(y). \quad (1)$$

Let $x \neq y$ and take f as in (**??**). Let $0 < \epsilon < |f(x) - f(y)|/2$ and let $N_x = \{z : |f(z) - f(x)| < \epsilon\}, N_y = \{z : |f(z) - f(y)| < \epsilon\}$. Then $x \in N_x, y \in N_y, N_x \cap N_y = \phi$, and $N_x, N_y \in \mathcal{T}$ so $\mathcal{T}$ is Hausdorff.

Conversely suppose (**??**) fails, that is, there exist x, y with $f(x) = f(y)$ for all $f \in \mathcal{F}$. Let $U \in \mathcal{T}$ be a neighborhood of x. Then $U \supset \cap_{i=1}^{n} \{z : |f_i(z) - f_i(x)| < \epsilon\} \ni x$ for some $n, f_1, .., f_n, \epsilon$, since such sets form a base. But this shows $y \in U$. Thus $\mathcal{T}$ is not Hausdorff.

(34) Suppose $\langle x_\alpha \rangle \to x$ in the weak topology generated by $\mathcal{F}$. Let $f \in \mathcal{F}$ and let U be a neighborhood of $f(x)$. Then $f^{-1}(U)$ is (by definition of the weak topology) a neighborhood of x so $x_\alpha \in f^{-1}(U)$ eventually, that is, $f(x_\alpha) \in U$ eventually. This shows that $\langle f(x_\alpha) \rangle \to f(x)$.

Conversely suppose $\langle f(x_\alpha) \rangle \to f(x)$ for all $f \in \mathcal{F}$. Let V be a neighborhood of x. Then $V \supset W = \cap_{i=1}^{n} f_i^{-1}(U_i) \ni x$ for some $f_1, .., f_n \in \mathcal{F}$ and open sets $U_1, .., U_n$, since such sets W form a base. It follows from the fact that $\langle f_i(x_\alpha) \rangle \to f_i(x)$ for each i that we have $f_i(x_\alpha) \in U_i$ for each $i \leq n$, so $x_\alpha \in W$ eventually. This shows that $\langle x_\alpha \rangle \to x$.

(54)(a) Let U be open in $\mathbb{Q}$; then $U = W \cap \mathbb{Q}$ for some open W in $\mathbb{R}$. We will show $\overline{U}$ is not compact. Let I be any one of the open intervals comprising W and let $J = I \cap \mathbb{Q}$. Since

$\overline{J} \subset \overline{U}$, it is enough to show $\overline{J}$ is not compact. Let $\{q_n\}$ be a sequence in J converging to an irrational r. Then $\{q_n\}$ has no converging subsequence in $\overline{J}$, so $\overline{J}$ is not compact.

(b) Clearly $\mathbb{Q}$ is σ-compact, since each singleton $\{q\}$ is compact. Let $\{q_n\}$ be a sequence in $\mathbb{Q} \cap [0,1]$ with $q_n \to 1$. Claim: $K = \{q_n, n \geq 1\} \cup \{1\}$ is compact in $\mathbb{Q}$. Proof: In any open cover of K, some open set covers 1 and hence covers all but finitely many q_n's. Thus only finitely many more sets are needed to cover all of K, so K is compact. Let

$$f_n(x) = \begin{cases} x^n, & x \in [0,1] \\ 0, & \text{otherwise,} \end{cases} \qquad f(x) = \begin{cases} 0, & x \neq 1 \\ 1, & x = 1. \end{cases}$$

Then f_n converges pointwise to f, but not uniformly on K.

(58) Suppose infinitely many X_α are noncompact, and $K \subset \prod_{\alpha \in A} X_\alpha$ is compact (hence closed.) We want to show that K contains no open set, so K is nowhere dense. Let U be open; then U contains a base set V, which for the product topology has the form $V = \prod_{i=1}^n U_{\alpha_i} \times \prod_{\alpha \in \tilde{A}} X_\alpha$, with $\tilde{A} = A \backslash \{\alpha_1, .., \alpha_n\}$ and with U_{α_i} open. Then $\overline{V} = \prod_{i=1}^n \overline{U_{\alpha_i}} \times \prod_{\alpha \in \tilde{A}} X_\alpha$. Since infinitely many X_α are noncompact, X_{α_0} must be noncompact for some $\alpha_0 \in \tilde{A}$, meaning there exists an open cover $\{G_\beta, \beta \in B\}$ of X_{α_0} with no finite subcover. But then $\{G_\beta \times \prod_{\alpha \neq \alpha_0} X_\alpha : \beta \in B\}$ is an open cover of $\overline{V}$ with no finite subcover, so $\overline{V}$ is not compact. Since $\overline{V}$ is closed, it cannot then be a subset of K, so $V \not\subset K$, so $U \not\subset K$. This shows that K is nowhere dense.

(A) Suppose $f_n \to f$ uniformly on compact subsets of X. Then for each fixed m, as $n \to \infty$, we have $\sup_{x \in \overline{U_m}} |f_n(x) - f(x)| \to 0$ and hence $\Phi(\sup_{x \in \overline{U_m}} |f_n(x) - f(x)|) \to 0$. Let $\epsilon > 0$ and choose k so that $\sum_{m=k+1}^\infty 2^{-m} = 2^{-k} < \epsilon/2$, and choose N such that $n \geq N$ implies $\Phi(\sup_{x \in \overline{U_m}} |f_n(x) - f(x)|) < \epsilon/2k$ for all $m \leq k$. Then since $|\Phi| \leq 1$, $n \geq N$ implies

$$\begin{aligned} \rho(f_n, f) &\leq \sum_{m=1}^k \Phi(\sup_{x \in \overline{U_m}} |f_n(x) - f(x)|) + \sum_{m=k+1}^\infty 2^{-m} \\ &< k\frac{\epsilon}{2k} + \frac{\epsilon}{2} \\ &= \epsilon. \end{aligned}$$

This shows that $\rho(f_n, f) \to 0$.

Conversely suppose $\rho(f_n, f) \to 0$. Since the terms in the series defining $\rho(f_n, f)$ are nonnegative, each one must approach 0, so $\Phi\left(\sup_{x \in \overline{U_m}} |f(x) - f_n(x)|\right) \to 0$ as $n \to \infty$, for all m, so also $\sup_{x \in \overline{U_m}} |f(x) - f_n(x)| \to 0$ as $n \to \infty$, that is, the convergence is uniform over each $\overline{U_m}$. But given a compact K, the sets $U_1, U_2, ...$ form an open cover for K since their union is X, so there is a finite subcover: for some M, $K \subset \cup_{m=1}^M U_m = U_M$. Since the covergence is uniform over $\overline{U_M}$ it is uniform over K. Thus $f_n \to f$ uniformly on compact sets