

MATH 525b ASSIGNMENT 3 SOLUTIONS  
 SPRING 2009  
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Chapter 5

(22)(a) Let  $T \in L(\mathcal{X}, \mathcal{Y})$  and define  $T^\dagger : \mathcal{Y}^* \rightarrow \mathcal{X}^*$  by  $T^\dagger f = f \circ T$ . Clearly  $T^\dagger$  is linear. Also

$$\begin{aligned} \|T^\dagger f\| &= \sup\{|(T^\dagger f)(x)| : x \in \mathcal{X}, \|x\| = 1\} \\ &= \sup\{|f(Tx)| : x \in \mathcal{X}, \|x\| = 1\} \\ &\leq \sup\{\|f\| \|Tx\| : x \in \mathcal{X}, \|x\| = 1\} \\ &= \|f\| \|T\|, \end{aligned}$$

so  $T^\dagger \in L(\mathcal{Y}^*, \mathcal{X}^*)$ . This also shows that  $\|T^\dagger\| \leq \|T\|$ .

For the reverse inequality, let  $\epsilon > 0$ . By (5.3) there exists  $x \neq 0$  in  $\mathcal{X}$  with  $\|Tx\|/\|x\| \geq (1 - \epsilon)\|T\|$ . By Theorem 5.8b there exists  $f \in \mathcal{Y}^*$  with  $\|f\| = 1$  and  $f(Tx) = \|Tx\|$ . Then

$$|(T^\dagger f)(x)| = |f(Tx)| = \|Tx\| \geq (1 - \epsilon)\|T\| \|x\|,$$

so

$$\|T^\dagger f\| = \sup\left\{\frac{|(T^\dagger f)(y)|}{\|y\|} : y \neq 0\right\} \geq (1 - \epsilon)\|T\|.$$

Thus

$$\|T^\dagger\| = \sup\{\|T^\dagger f\| : \|f\| = 1\} \geq (1 - \epsilon)\|T\|.$$

Since  $\epsilon > 0$  is arbitrary, this shows  $\|T^\dagger\| \geq \|T\|$ , so they are equal.

(47)(a) Suppose  $\{T_n\} \subset L(X, Y)$  and  $T_n \rightarrow T$  weakly, i.e. in the weak operator topology. (This includes the case of  $T_n \rightarrow T$  strongly, since strong convergence implies weak convergence.) For  $f \in Y^*$  let  $\mathcal{A}_f = \{f \circ T_n : n \geq 1\}$ . Since  $T_n \rightarrow T$  weakly, for each  $x$  we have  $(f \circ T_n)(x) \rightarrow (f \circ T)(x)$ , so  $\sup_n \|(f \circ T_n)(x)\| < \infty$ . By 5.13b (applied to  $\mathcal{A}_f$ ), for each  $f \in Y^*$ ,  $\sup_n \|f \circ T_n\| < \infty$ , which is the same as  $\sup_n \|T_n^\dagger f\| < \infty$ . Applying 5.13b to  $\{T_n^\dagger : n \geq 1\}$ , we get  $\sup_n \|T_n^\dagger\| < \infty$ . By problem 22a, this is the same as  $\sup_n \|T_n\| < \infty$ .

(b) If  $x_n \rightarrow x$  weakly in  $X$ , then (essentially by definition, since the weak and strong topologies are the same on  $\mathbb{R}$ )  $\hat{x}_n \rightarrow \hat{x}$  in the weak operator topology on  $L(X^*, \mathbb{R})$ . Hence by (a),  $\sup_n \|\hat{x}_n\| < \infty$ . But  $\|x_n\| = \|\hat{x}_n\|$  so  $\{x_n\}$  is bounded.

If  $f_n \rightarrow f$  in the weak\* topology on  $X^*$ , that is,  $f_n(x) \rightarrow f(x)$  for all  $x \in X$ , then  $\sup_n |f_n(x)| < \infty$  for all  $n$ , so by 5.13b,  $\sup_n \|f_n\| < \infty$ , that is,  $\{f_n\}$  is bounded.

(51) Let  $Y$  be a vector subspace of  $X$ . If  $Y$  is weakly closed,  $\{x_n\} \subset Y$  and  $\|x_n - x\| \rightarrow 0$ , then  $x_n \rightarrow x$  weakly so  $x \in Y$ . Thus  $Y$  is norm-closed.

Conversely suppose  $Y$  is norm-closed,  $\{x_n\} \subset Y$  and  $x_n \rightarrow x$  weakly. If  $x \notin Y$  then by 5.8a, there exists  $f \in X^*$  with  $f = 0$  on  $Y$  (so  $f(x_n) = 0$ ) and  $f(x) \neq 0$ , which means  $f(x_n) \not\rightarrow f(x)$ , contradicting  $x_n \rightarrow x$  weakly. Thus  $x \in Y$ , so  $Y$  is weakly closed.

(53) Suppose  $T_n \rightarrow T$  and  $S_n \rightarrow S$ , both strongly, in  $L(X, X)$ .

(a) Suppose  $\{x_n\} \subset X$  and  $\|x_n - x\| \rightarrow 0$ . By Exercise 47a,  $M = \sup_n \|T_n\| < \infty$ . Hence

$$\|T_n x_n - Tx\| \leq \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leq M\|x_n - x\| + \|T_n x - Tx\|.$$

The first term on the right approaches 0 by assumption, and the second term approaches 0 since  $T_n \rightarrow T$  strongly, so  $\|T_n x_n - Tx\| \rightarrow 0$ .

(b) For each  $x$  we have  $\|S_n x - Sx\| \rightarrow 0$ , so we can apply (a) (to  $S_n x, Sx$  in place of  $x_n, x$ ) to get  $\|T_n S_n x - TSx\| \rightarrow 0$ . This shows that  $T_n S_n \rightarrow TS$  strongly.

(56) Let  $E$  be a subset of a Hilbert space. If  $x \in E$  and  $y \in E^\perp$ , then  $\langle x, y \rangle = 0$ , which shows that  $x \in (E^\perp)^\perp$ . Thus  $E \subset (E^\perp)^\perp$ , and (from the text before 5.23)  $(E^\perp)^\perp$  is a closed subspace.

Now let  $\mathcal{M}$  be a closed subspace containing  $E$ . Then  $\mathcal{M}^\perp \subset E^\perp$  so  $(\mathcal{M}^\perp)^\perp \supset (E^\perp)^\perp$ . We claim that  $(\mathcal{M}^\perp)^\perp = \mathcal{M}$ . To prove this, note that from the first paragraph with  $\mathcal{M}$  in place of  $E$ , we have  $\mathcal{M} \subset (\mathcal{M}^\perp)^\perp$ . If these are not equal, then there exists  $y \neq 0$  in  $(\mathcal{M}^\perp)^\perp$  orthogonal to  $\mathcal{M}$ , that is,  $y \in \mathcal{M}^\perp$ . But then  $y \perp y$  so  $y = 0$ . Thus the claim is proved, so  $\mathcal{M} = (\mathcal{M}^\perp)^\perp \supset (E^\perp)^\perp$ , that is,  $(E^\perp)^\perp$  is smaller than  $\mathcal{M}$ . Thus  $(E^\perp)^\perp$  is the smallest closed subspace containing  $E$ .

(57)(a) Let  $V : \mathcal{H} \rightarrow \mathcal{H}^*$  be given by  $(Vy)(x) = \langle x, y \rangle$  for  $x, y \in \mathcal{H}$ . Then by Theorem 5.25,  $V$  is a conjugate-linear norm-preserving bijection. By definition of  $V$ , for  $f \in \mathcal{H}^*$  we have  $f(x) = \langle x, V^{-1}f \rangle$ . Let  $T \in L(\mathcal{H}, \mathcal{H})$  and  $T^* = V^{-1}T^\dagger V$ , which maps  $\mathcal{H}$  to  $\mathcal{H}$ . (Here  $T^\dagger$  is from problem 22a.) Then for all  $x, y \in \mathcal{H}$ , applying the preceding to  $f = T^\dagger Vy$  we get

$$\langle Tx, y \rangle = (Vy)(Tx) = (T^\dagger(Vy))(x) = \langle x, V^{-1}T^\dagger Vy \rangle = \langle x, T^*y \rangle.$$

$T^*$  is linear since for all  $x$ ,

$$\begin{aligned} \langle x, T^*(\lambda y_1 + y_2) \rangle &= \langle Tx, \lambda y_1 + y_2 \rangle \\ &= \bar{\lambda} \langle Tx, y_1 \rangle + \langle Tx, y_2 \rangle \\ &= \bar{\lambda} \langle x, T^*y_1 \rangle + \langle x, T^*y_2 \rangle \\ &= \langle x, \lambda T^*y_1 + T^*y_2 \rangle, \end{aligned}$$

which shows that  $T^*(\lambda y_1 + y_2) = \lambda T^*y_1 + T^*y_2$ .  $T^*$  is unique since if there were another operator  $U$  with the same property, we would have  $\langle x, T^*y \rangle = \langle Tx, y \rangle = \langle x, Uy \rangle$  for all  $x, y$ ,

and hence  $\langle x, T^*y - Uy \rangle = 0$  for all  $x, y$ , and hence  $T^*y - Uy = 0$  for all  $y$ , which says  $T^* = U$ . Boundedness of  $T$  is proved in (b) below.

(b) Since  $V$  is a norm-preserving bijection, we have

$$\|T^*\| = \sup_{\|x\|=1} \|T^*x\| = \sup_{\|x\|=1} \|VT^*x\| = \sup_{\|x\|=1} \|T^\dagger Vx\| = \sup_{\|y\|=1} \|T^\dagger y\| = \|T^\dagger\| = \|T\|,$$

the third equality following from the definition of  $T^*$  and the last equality following from Exercise 22a. Next, for every  $x$  with  $\|x\| = 1$ , we have

$$\|T^*Tx\| = \sup_{\|y\|=1} \langle y, T^*Tx \rangle = \sup_{\|y\|=1} \langle Ty, Tx \rangle \geq \langle Tx, Tx \rangle = \|Tx\|^2.$$

Thus, taking the sup of this over  $x$ ,

$$\sup_{\|x\|=1} \|T^*Tx\| \geq \left( \sup_{\|x\|=1} \|Tx\| \right)^2 = \|T\|^2,$$

so  $\|T^*T\| \geq \|T\|^2$ . Also  $\|T^*T\| \leq \|T^*\| \|T\| = \|T\|^2$  by the preceding, so  $\|T^*T\| = \|T\|^2$ .

Next, for all  $x, y$ ,

$$\begin{aligned} \langle x, (aS + bT)^*y \rangle &= \langle (aS + bT)x, y \rangle \\ &= a\langle Sx, y \rangle + b\langle Tx, y \rangle \\ &= a\langle x, S^*y \rangle + b\langle x, T^*y \rangle \\ &= \langle x, (\bar{a}S^* + \bar{b}T^*)y \rangle \end{aligned}$$

so  $(aS + bT)^*y = (\bar{a}S^* + \bar{b}T^*)y$  for all  $y$ , which means that  $(aS + bT)^* = \bar{a}S^* + \bar{b}T^*$ .

Next, for all  $x, y$ ,

$$\langle x, (ST)^*y \rangle = \langle STx, y \rangle = \langle Tx, S^*y \rangle = \langle x, T^*S^*y \rangle,$$

so  $(ST)^*y = T^*S^*y$  for all  $y$ , which says that  $(ST)^* = T^*S^*$ .

Finally, for all  $x, y$ ,

$$\langle x, (T^*)^*y \rangle = \langle T^*x, y \rangle = \overline{\langle y, T^*x \rangle} = \overline{\langle Ty, x \rangle} = \langle x, Ty \rangle,$$

so as above,  $(T^*)^* = T$ .

(63)(a) Let  $\mathcal{H}$  be infinite-dimensional and let  $\{u_n\}$  be an orthonormal sequence in  $\mathcal{H}$ . For every  $x$  we have  $\infty > \|x\|^2 \geq \sum_{n=1}^{\infty} |\langle x, u_n \rangle|^2$  by Bessel's inequality, so  $\langle x, u_n \rangle \rightarrow 0$ . This shows that  $u_n \rightarrow 0$  weakly.