MATH 525b ASSIGNMENT 4
SPRING 2009
Prof. Alexander
Due Monday March 9.

Chapter 6 #3, 7, 8, 12, 22b and:

(I) Is it true that for Lebesgue measure m on $(0, \infty)$,

$$\bigcap_{1 \le p < \infty} L^p(m) = L^\infty(m) \ ?$$

Justify your answer.

(II) Let m be Lebesgue measure on $[0, 1]$ and let $1 < p < \infty$, and suppose $f \in L^p(m)$. Show that

$$\lim_{y \searrow 0} y^{-(p-1)/p} \int_{[0,y]} f \ dm = 0.$$

(III) Let $0 < p < 1$. let m be Lebesgue measure on $(0, 1)$ and define distance on $L^p(m)$ by

$$d_p(f, g) = \int_{[0,1]} |f - g|^p \ dm.$$

(You may take as given that this is a metric. Note we do not take the $1/p$ power, as this does not give a metric for $0 < p < 1$.) Define $F : (0, 1) \to L^p(m)$ by $F(t) = \chi_{[0,t]}$. Show that $F'(t) = 0$ even though F is not constant. Here F' is defined as the limit, for the distance d_p, of the difference quotients.

Just as a note, this can be viewed as a consequence of having a topology that is not locally convex.

HINTS:

(3) The only tricky part of showing $L^p \cap L^r$ is a Banach space is showing that it is complete. Showing the inclusion map is continuous means showing that if $f_n \to f$ in $L^p \cap L^r$, then $f_n \to f$ in L^q.

(7) One part is to show that given $\epsilon > 0$, for sufficiently large q we have $||f||_q \ge ||f||_\infty - \epsilon$. To see how to do this, contemplate this idea: the statement that you want to prove, that $\lim_{q\to\infty} ||f||_q = ||f||_\infty$, means that $\lim_{q\to\infty} ||f||_q$ is not affected if the function is changed wherever $|f|$ is not close to its essential sup. Of course you can't use the statement to prove

itself, but it may suggest the right idea.

(8)(a) You may assume Exercise 42d.

(b) This one is complicated! (So if you *have* to skip something, skip this.) You will need certain estimates which can be obtained from the Taylor Series for e^x:

$$\frac{e^x - 1}{x} < e^x \quad \text{for } x > 0, \qquad 1 - e^{-x} \leq x \quad \text{for } x \geq 0.$$

You will also need Fatou's Lemma, Dominated Convergence, and more.

(c) Use (a) and (b).

(12) To show the parallelogram law fails, you just need one example of two functions where this happens. So try to find one using a particularly easy-to-work-with type of function.