MATH 525b ASSIGNMENT 4 SOLUTIONS
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Chapter 6

(3) Let $1 \leq p < r \leq \infty$. Clearly $L^p \cap L^r$ is a normed linear space, so we need to verify completeness. Suppose $\{f_n\}$ is Cauchy in $L^p \cap L^r$. Then $\{f_n\}$ is Cauchy as a sequence in L^p, and as a sequence in L^r. Hence there exist $f, \tilde{f}$ such that $f_n \to f$ in L^p and $f_n \to \tilde{f}$ in L^r. Then f_n converges to both f and $\tilde{f}$ in measure, so by 2.30, $f = \tilde{f}$ a.e. This means $f_n \to f$ in $L^p \cap L^r$, so $L^p \cap L^r$ is complete.

Now let $p < q < r$ and suppose $f_n \to f$ in $L^p \cap L^r$. By Proposition 6.10, $||f_n - f||_q \leq ||f_n - f||_p^\lambda \, ||f_n - f||_r^{1-\lambda}$ where λ is given by $\frac{1}{q} = \frac{\lambda}{p} + \frac{1-\lambda}{r}$. Therefore $||f_n - f||_q \to 0$, so the inclusion map $L^p \cap L^r \top L^q$ is continuous.

(7) Let $f \in L^p \cap L^\infty$. Let $\lambda = p/q$, so $\lambda \to 0$ as $p \to \infty$. For all $q > p$, using Proposition 6.10, it follows that

$$||f||_q \leq ||f||_p^\lambda \, ||f||_\infty^{1-\lambda} \to ||f||_\infty \quad \text{as } q \to \infty (\text{i.e. as } \lambda \to 0),$$

so $\limsup_q ||f||_q \leq ||f||_\infty$.

For the opposite direction, let $\epsilon > 0$ and $E = \{x : |f(x)| > ||f||_\infty - \epsilon\}$. Then (assuming $\epsilon < ||f||_\infty$) we have $\mu(E) < \infty$ since $||f||_p < \infty$, and

$$\begin{aligned} ||f||_q &\geq \left(\int_E |f|^q \, d\mu \right)^{1/q} \\ &\geq \left((||f||_\infty - \epsilon)^q \mu(E) \right)^{1/q} \\ &= (|f||_\infty - \epsilon) \, \mu(E)^{1/q} \\ &\to ||f||_\infty - \epsilon \text{ as } q \to \infty. \end{aligned}$$

Since ϵ is arbitrary this shows $\liminf ||f||_q \geq ||f||_\infty$, so $\liminf ||f||_q = ||f||_\infty$.

(8)(a) Suppose $\mu(X) = 1$ and $f \in L^p$. Then $f \in L^q$ for all $q < p$. We may assume $\int \log |f| \, d\mu > -\infty$, for otherwise there is nothing to prove. $F(x) = -\log x$ is convex on $(0, \infty)$ so by Jensen's Inequality (Exercise 42d in Chapter 3), $-\log \int |f|^q \, d\mu \leq \int -\log |f|^q \, d\mu$, which after taking negatives is the same as $\log ||f||_q \geq \int \log |f| \, f\mu$.

(b) By convexity of e^x, for all x we have $e^x \geq 1 + x$, since $1 + x$ is the tangent line to e^x at $x = 0$. Taking $x = q \log ||f||_q$ we get $\int |f|^q \geq 1 + q \log ||f||_q$ or equivalently

$$\frac{\int |f|^q - 1}{q} \geq \log ||f||_q.$$

Now consider $q \to 0$, in particular $q < p/2$. For $x > 0$ we have

$$\frac{e^x - 1}{x} = \sum_{n=0}^{\infty} \frac{x^n}{(n+1)!} < e^x.$$

Taking $x = q \log |f|$ we see that when $|f| > 1$, we have

$$\frac{|f|^q - 1}{q} < (\log |f|)\ |f|^q \le (\log |f|)\ |f|^{p/2} \le c|f|^p, \tag{1}$$

where c is some constant (not depending on the value of f.) Also, for $x \ge 0$ we have $1 - e^{-x} \le x$ (by convexity of e^{-x}) so when $|f| \le 1$ we have

$$0 \le \frac{1 - |f|^q}{q} \le -\log |f|. \tag{2}$$

Let $E = \{t : |f(t)| \le 1\}$. By easy calculus,

$$\frac{|f|^q - 1}{q} \to \log |f| \quad \text{pointwise.}$$

By Fatou's Lemma,

$$\liminf_{q \to 0} \int_E \frac{1 - |f|^q}{q} \ge \int_E -\log |f|,$$

so

$$\limsup_{q \to 0} \int_E \frac{|f|^q - 1}{q} \le \int_E \log |f|. \tag{3}$$

By (2),

$$\liminf_{q \to 0} \int_E \frac{|f|^q - 1}{q} \ge \int_E \log |f|,$$

which with (3) shows

$$\lim_{q \to 0} \int_E \frac{|f|^q - 1}{q} = \int_E \log |f|.$$

For E^c, we can use (1) and Dominated Convergence to conclude

$$\lim_{q \to 0} \int_{E^c} \frac{|f|^q - 1}{q} = \int_{E^c} \log |f|.$$

The last two limits show that

$$\lim_{q \to 0} \int \frac{|f|^q - 1}{q} = \int \log |f|.$$

(c) By (a), $||f||_q \geq \exp(\int \log |f|)$. Given $\delta > 0$, by (b), when q is small enough,

$$||f||_q^q = \int |f|^q \leq 1 + (1+\delta)q \int \log |f| \leq \exp\left((1+\delta)q \int \log |f|\right),$$

so $||f||_q \leq \exp\left((1+\delta) \int \log |f|\right)$. Since δ is arbitrary, this shows $\lim_{q \to 0} ||f||_q = \exp\left(\int \log |f|\right)$.

(12) Let $p \neq 2$ and suppose that for some measure space, $\dim(L^p) > 1$. Then the σ-algebra must include two disjoint sets E, F with $0 < \mu(E) < \infty$, $0 < \mu(F) < \infty$ (we can take $F = E^c$.) Suppose the parallelogram identity holds for χ_E and χ_F, that is,

$$||\chi_E + \chi_F||_p^2 + ||\chi_E - \chi_F||_p^2 = 2\left(||\chi_E||_p^2 + ||\chi_F||_p^2\right).$$

This is equivalent to

$$(\mu(E) + \mu(F))^{2/p} = \mu(E)^{2/p} + \mu(F)^{2/p}. \tag{4}$$

If $p < 2$, the function $f(x) = (\mu(E) + x)^{2/p} - x^{2/p}$ is strictly increasing, and if $p > 2$ it is strictly decreasing. Either way we cannot have $f(\mu(F)) = f(0)$, which is equivalent to (4). Thus $p = 2$.

(22)(b) Let $X = [0, 1]$ and $f_n = n\chi_{(0,1/n)}$. Clearly $f_n(x) \to 0$ a.e., and $\mu(\{x : f_n(x) \neq 0\}) = 1/n \to 0$, so $f_n \to 0$ in measure.

Let $p > 1$ and let q be its conjugate. Let $\alpha < 1/q$ and $g(x) = x^{-\alpha}$, so $g \in L^q$. Then for the linear functional $\varphi_g(f) = \int fg$,

$$\varphi_g(f_n) = n \int_0^{1/n} x^{-\alpha}\, dx = \frac{n^\alpha}{1-\alpha} \to \infty \qquad \text{as } n \to \infty,$$

so $f_n \not\to 0$ weakly in L^p.

For $p = 1$, let $h(x) \equiv 1$. Then $h \in L^\infty = (L^1)^*$, and $\varphi_h(f_n) = 1$ for all n, so $f_n \not\to 0$ weakly in L^1.

(I) No. If a function $f(x) \to \infty$ as $x \to 0$ but more slowly than any power of x, then it will be in L^p for all $1 \leq p < \infty$ but not in L^∞. For example, this is true for

$$f(x) = \begin{cases} \log \frac{1}{x}, & x \in (0, 1) \\ 0, & x \in [1, \infty). \end{cases}$$

(II) Let q be conjugate to p. By Hölder's Inequality we have

$$\begin{aligned} y^{-(p-1)/p} \left| \int_{[0,y]} f \; dm \right| &= y^{-1/q} \left| \int f\chi_{[0,y]} \cdot \chi_{[0,y]} \; dm \right| \\ &\leq y^{-1/q} \|f\chi_{[0,y]}\|_p \|\chi_{[0,y]}\|_q \\ &= \|f\chi_{[0,y]}\|_p \\ &= \left(\int |f|^p \chi_{[0,y]} \; dm \right)^{1/p} \\ &\to 0 \quad \text{as } y \searrow 0, \end{aligned}$$

since $|f|^p$ is integrable.

(III) We have for $h > 0$:

$$\begin{aligned} d_p\left(\frac{F(t+h) - F(t)}{h}, 0 \right) &= \int_{[0,1]} \left(\frac{1}{h} \chi_{[t,t+h]} \right)^p \; dm \\ &= h^{1-p} \\ &\to 0 \quad \text{as } h \searrow 0. \end{aligned}$$

A similar argument applies for $h < 0$, with $\chi_{[t-h,t]}$ in place of $\chi_{[t,t+h]}$. Thus $F'(t) = 0$ for all t.