MATH 525b ASSIGNMENT 5
SPRING 2009
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Due Wednesday April 1.

Chapter 7 #17, 20a, 22 and the problems below.

(A) An operator $P \in L(X, X)$ is called a projection if $P^2 = P$.
(a) Show that $\mathcal{R}(P) = \{x : Px = x\}$, so $\mathcal{R}(P)$ is a closed subspace.
(b) Show that the following are equivalent:

(i) P is self-adjoint;

(ii) P is normal;

(iii) $\mathcal{R}(P) = \mathcal{N}(P)^\perp$.

(B) Let $T \in L(X, X)$.
(a) Show that $\langle Tx, x\rangle = 0$ for all x implies $T = 0$.
(b) Show that if $||Tx|| = ||T^*x||$ for all x, then T is normal.

(C) Let $T \in L(X, X)$. If $\langle Tx, x\rangle \geq 0$ for all x, show that T is self-adjoint and $\sigma(T) \subset [0, \infty)$.

(D) Suppose X is a separable Hilbert space and $T \in L(X, X)$ is normal. Show that $\sigma_p(T)$ is at most countable.

(E)(Harder problem) Let $X = \ell^2 = \{(x_0, x_1, ..) : x_i \in \mathbb{C}, \sum |x_i|^2 < \infty\}$ and define the left and right shifts S_L, S_R by

$$S_L(x_0, x_1, ...) = x_1, x_2, ...), \qquad S_R(x_0, x_1, ...) = (0, x_1, x_2, ...).$$

Show that

(i) $S_R^* = S_L$;

(ii) $\sigma_p(S_L) = \sigma_r(S_R) = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$;

(iii) $\sigma_c(S_L) = \sigma_c(S_R) = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$,

(iv) $\sigma_r(S_L) = \sigma_p(S_R) = \phi$.

HINTS:

(20)(a) Reduce the problem to showing that there is no $f \in C_0(X)$ with the property that the sum of the point masses in μ is equal to $\int f \, d\mu$ for all finite Radon μ. You are assuming there is a nonzero measure μ_0 with no point masses; what other measures derived from μ_0 also have no point masses?

(A) To show (iii) implies (i), first show that $\langle Px, y\rangle = \langle Px, Py\rangle = \langle x, Py\rangle$ for all x, y.

(B)(a) Given $x, y \in X$, you can apply the assumption to $x + \lambda y$ in place of x, for whatever value(s) of λ are useful.

(b) For "if", use (a).

(C) Let $\lambda > 0$; you want to show that $-\lambda \notin \sigma(T)$. Show that $||(\lambda I + T)x|| \geq \lambda ||x||$, by considering properties of $\langle \lambda x, x\rangle$.

(E) For proving (ii)–(iv) (which are really 6 facts), the order you prove things is important because some of the 6 statements make use of the others in their proofs. Here is a suggested order to establish things: $\sigma_p(S_L), \sigma_p(S_R), \sigma_r(S_L), \sigma_r(S_R), \sigma_c(S_L), \sigma_c(S_R)$.