

MATH 525b ASSIGNMENT 6 SOLUTIONS
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(I) Let $E(\cdot)$ be a resolution of the identity corresponding to T , so $T = \int_{\sigma(T)} \lambda E(d\lambda)$ and $T^* = \int_{\sigma(T)} \bar{\lambda} E(d\lambda)$. When T is invertible (that is, $0 \notin \sigma(T)$), we have $T^{-1} = \int_{\sigma(T)} \lambda^{-1} E(d\lambda)$.

In general, for $Y = \sigma(T)$ and $f, g \in L^2(Y, \mu)$, if $\int_{\sigma(T)} f(\lambda) E(d\lambda) = \int_{\sigma(T)} g(\lambda) E(d\lambda)$ (that is, $f(T) = g(T)$), then by unitary equivalence, $M_f = M_g$ as operators on $L^2(Y, \mu)$. Since $\mu(A) > 0$ for all open sets A which intersect $\sigma(T)$, $M_f = M_g$ implies $\int (f - g)h d\mu = 0$ for all $h \in L^2(Y, \mu)$, which implies $f = g$ a.e. If f, g are continuous, this implies $f = g$ everywhere.

In particular we have the following equivalence:

$$\begin{aligned}
 T \text{ unitary} &\iff T^* = T^{-1} \\
 &\iff \lambda^{-1} = \bar{\lambda} \text{ for all } \lambda \in \sigma(T) \\
 &\iff |\lambda|^2 = 1 \text{ for all } \lambda \in \sigma(T) \\
 &\iff \sigma(T) \text{ is contained in the unit circle.} \tag{1}
 \end{aligned}$$

(II) Since $(f_n - f)(T)$ is normal, we have

$$\|f_n(T) - f(T)\| = \rho((f_n - f)(T)) = \sup_{x \in \sigma(T)} |(f_n - f)(x)| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Here the first inequality is from a proposition from lecture, and the second equality is from the Spectral Mapping Theorem.

(III) Write $z \in \mathbb{C}$ as $re^{i\theta}$ with $\theta \in [0, 2\pi)$. Define $f(re^{i\theta}) = r^{1/n}e^{i\theta/n}$. Then f is locally bounded and $f(z)^n = z$. Therefore $f(T)^n = T$.

Now $z \neq 0 \iff f(z) \neq 0$, so if T is invertible, then $0 \notin \sigma(T)$ so (using the Spectral Mapping Theorem) $0 \notin f(\sigma(T)) = \sigma(f(T))$, so $S = f(T)$ is invertible.

(IV) Define $f(\lambda) = \bar{\lambda}/\lambda$ for $\lambda \neq 0$, and $f(0) = 1$. Let $Y = \sigma(T)$. Since f is bounded we can define $U = f(T)$. Let $g(\lambda) = \bar{\lambda}$. By the Spectral Mapping Theorem, $\sigma(U) = \{f(\lambda) : \lambda \in \sigma(T)\}$ and $|f(\lambda)| = 1$ for all λ , so by problem (I), U is unitary. Since $f(\lambda)\lambda = \bar{\lambda} = g(\lambda)$, we have $UT = f(T)T = g(T) = T^*$.

(V)(a) Suppose T is normal and $\overline{\mathcal{R}(\lambda I - T)} \neq \mathcal{X}$. Then $\mathcal{N}(\lambda I - T) = \mathcal{N}((\lambda I - T)^*) = \overline{\mathcal{R}(\lambda I - T)}^\perp \neq \phi$, so $\lambda \in \sigma_p(T)$. This shows that $\sigma_r(T) = \phi$.

(b) Suppose $\|\psi\|_\infty < \infty$. For all λ we have $(\lambda I - M_\psi)f = (\lambda - \psi)f$ for all $f \in L^2$. Therefore if $\lambda I - M_\psi$ has an inverse, this inverse is $M_{1/(\lambda - \psi)}$. Hence we have the

following equivalence: $\lambda \notin \sigma(T) \iff M_{\lambda-\psi}$ is invertible $\iff M_{1/(\lambda-\psi)}$ is bounded $\iff \|1/(\lambda-\psi)\|_\infty < \infty \iff$ except on a null set, ψ is bounded away from λ . The last statement can be stated more precisely as “ $\mu(\{x : |\psi(x) - \lambda| < \epsilon\}) = 0$ for some $\epsilon > 0$.”