MATH 525b ASSIGNMENT 7
SPRING 2009
Prof. Alexander
Due Wednesday April 29.

(A) Let F be a monotone nondecreasing function on $\mathbb{R}$. Show that its distributional derivative ∂F is a Borel measure, that is, $(\partial F)(\varphi) = \int \varphi \, d\mu$ for $\varphi \in \mathcal{D}(\mathbb{R})$. HINT: Modify F so it is right-continuous; it is then the distribution function of a measure.

(B) Let $\mathcal{N}(T) \subset \mathcal{D}(\Omega)$ be the nullspace of the distribution T. Show that there is a function $\varphi_0 \in \mathcal{D}(\Omega)$ such that every $\varphi \in \mathcal{D}(\Omega)$ has the form $\varphi = c\varphi_0 + \psi$ with $\psi \in \mathcal{N}(T), c \in \mathbb{C}$. HINT: This is really just a general fact about linear functionals, specified to the case of distributions.

(C) Show that $f \in W^{1,\infty}(\mathbb{R}^n)$ if and only if f is (in the sense of $W^{1,\infty}(\mathbb{R}^n)$) a Lipschitz-continuous function. Here "in the sense of $W^{1,\infty}(\mathbb{R}^n)$" means $f = g$ a.e. for some such g, and Lipschitz-continuous means there exists c such that $|g(y) - g(x)| \leq c|y - x|$ for all x, y. HINT: Use Corollary 6 from lecture. Also, by Fubini, if for each y a statement is true for a.e. x, then for a.e. x the statement is true for a.e. y.

(D) Let $f(x) = |x|^{-n}$ on $\mathbb{R}^n \backslash \{0\}$, and let T_f be the corresponding distribution on $\mathcal{D}(\mathbb{R}^n \backslash \{0\})$: $T_f(\varphi) = \int_{\mathbb{R}^n} \varphi(x)|x|^{-n} \, dx$. Note that $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Show, though, that there is an extension T of T_f to a distribution on $\mathcal{D}(\mathbb{R}^n)$, and give an explicit formula for T. HINT: If $\varphi(0) \neq 0$, then the original formula for T_f cannot be used because f is not integrable. Let $T(\varphi) = \int_{\mathbb{R}^n} \tilde{\varphi}(x)|x|^{-n} \, dx$, where $\tilde{\varphi}$ is some modification of φ that avoids this problem. You have to show that this new definition gives something continuous and linear.

(E) Suppose Ω is connected, $T \in \mathcal{D}'(\Omega)$, and for some m, $D^\alpha T = 0$ for all α with $|\alpha| = m+1$. Show that T is a polynomial of degree at most m (that is, show $T = T_f$ for some such polynomial f.) HINT: Induction on m.