MATH 525b TAKE-HOME MIDTERM EXAM
SPRING 2009
Prof. Alexander

What is allowed: Use of Folland and other published works (including information you did not request but you found already posted on the internet), your lecture notes, homework, and homework solutions.

What is not allowed: Use of nonpublished materials (e.g. someone else's lecture notes or solutions); soliciting or receiving help via the internet e.g. by posting a message on a forum; *consulting with each other or with anyone except me.* The latter is grounds for a 0 on the exam!

You can use all lemmas, theorems, etc. that were discussed in lecture or are found in Folland. If you use anything else, you need to include a proof.

Do *any 4 of the 6 problems.* You should turn in only the 4 you want graded. 25 points per problem.

The exam is due by Friday March 13 in lecture.

(1) Let X be the space of all sequences $\{a_n\}$ in $\mathbb{R}$ for which $\lim_{n\to\infty} a_n$ exists; we endow X with the uniform norm. Determine the nature of the bounded linear functionals on X. In other words, give a description of the dual which is specific to this X. HINT: First consider the subspace Y consisting of sequences satisfying $\lim_{n\to\infty} a_n = 0$. Let f be a bounded linear functional on Y, let $e_n = (0, \dots, 0, 1, 0, \dots)$ and let $b_n = f(e_n)$. Consider properties of $\{b_n\}$.

(2) Suppose $x_n \to x$ weakly in a Hilbert space H. Show that there exists a subsequence $\{x_{n_k}\}$ such that

$$\frac{1}{N}\sum_{k=1}^{N} x_{n_k} \to x \quad \text{strongly (i.e. in norm.)}$$

HINT: Reduce to the case $x = 0$. Also, $||x||^2 = \langle x, x\rangle$.

(3) Folland Chapter 5 #37, p. 165. HINT: This is quite short if you think about it correctly.

(4) Folland Chapter 6 #5, p. 186, first half only (about sets with arbitrarily small positive measure.)

(5) Show that every infinite-dimensional Banach space X has a subspace which is not closed.

HINT: X infinite dimensional means no finite set spans X, so one can inductively obtain $x_1, x_2, \dots$ with $x_{n+1} \notin \mathrm{span}(x_1, \dots, x_n)$ for all n. The tricky part is to deal with the following issue: if you take a vector of form $\sum_{n=1}^{\infty} a_n x_n$, and you want it not to be in $\mathrm{span}(x_1, \dots, x_n)$ for any n, you have to choose the a_n's in a way that avoids this.

You can use Exercise 18b Chapter 5 without proof, since it was assigned in 525a.

Also note that if $S_1 \subset S_2 \subset \dots$ are subspaces, then $\cup_{n \geq 1} S_n$ is a subspace.

(6) By Proposition 4.3, the collection of h-intervals $\{[a, b) : -\infty < a < b \leq \infty\}$ forms a base for a topology $\mathcal{T}_h$ on $\mathbb{R}$. (You may take this as given.)

(a) Show that $(\mathbb{R}, \mathcal{T}_h)$ is first countable but not second countable.

(b) Show that every compact set in $(\mathbb{R}, \mathcal{T}_h)$ is countable. HINT: What sequences converge in the usual topology on $\mathbb{R}$ but not in $(\mathbb{R}, \mathcal{T}_h)$? If K is compact and $x \in K$, what does this tell you about certain points near x?

(c) Give an example of an infinite compact set in $(\mathbb{R}, \mathcal{T}_h)$, with proof of compactness.