MATH 525b TAKE-HOME MIDTERM EXAM SOLUTIONS
SPRING 2009
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(1) For $x = (x_1, x_2, \dots) \in X$ let $L(x) = \lim_n x_n$, and let $Y = \{x \in X : L(x) = 0\}$. Let $f$ be a bounded linear functional on $Y$, let $b_n = f(e_n)$ and let $b = (b_1, b_2, \dots)$. For $y \in Y$ let $y^{(n)} = (y_1, \dots, y_n, 0, 0, \dots)$. Then $y^{(n)} \to y$ so

$$f(y) = \lim_n f(y^{(n)}) = \lim_n \sum_{k=1}^{n} b_k y_k = \sum_{k=1}^{\infty} b_k y_k. \tag{1}$$

We claim that $b \in l^1$. Suppose to the contrary that $\sum_{k=1}^{\infty} |b_k| = \infty$, and let $s_n = \sum_{k=1}^{n} |b_k|$, so $s_n \nearrow \infty$. For $y_k = s_k^{-1/2}\text{sign}(b_k)$ we have $y_k \to 0$ so $y \in Y$, but

$$\sum_{k=1}^{n} b_k y_k = \sum_{k=1}^{n} s_k^{-1/2} |b_k| \geq s_n^{-1/2} \sum_{k=1}^{n} |b_k| = s_n^{1/2} \to \infty,$$

so no bounded linear functional is given by formula (1). Thus $b \in l^1$ for every bounded linear functional $f$ on $Y$. Conversely if $b \in l^1$ then for $f$ given by (1) we have $|f(y)| \leq \sum_{k=1}^{\infty} |b_k| \ |y_k| \leq (\sum_{k=1}^{\infty} |b_k|) \, ||y||_\infty = ||b||_1 \ ||y||_\infty$ so $f(y) = \sum_{k=1}^{\infty} b_k y_k$ defines a bounded linear functional on $Y$.

Now let $u = (1, 1, \dots)$ and let $f$ be a bounded linear functional on $X$. By the above there exists $b \in l^1$ such that for $y \in Y$ we have $f(y) = \sum_{k=1}^{\infty} b_k y_k$. Let $a = f(u) - \sum_{k=1}^{\infty} b_k$. For $x \in X$ we have $x - L(x)u \in Y$ so

$$f(x) - L(x)f(u) = f(x - L(x)u) = \sum_{k=1}^{\infty} b_k (x_k - L(x))$$

so

$$f(x) = aL(x) + \sum_{k=1}^{\infty} b_k x_k. \tag{2}$$

Conversely for arbitrary $a \in \mathbb{R}$ and $b \in l^1$, (2) defines a bounded linear functional on $X$. Thus every bounded linear functional on $X$ has form (2).

(2) By subtracting $x$ from everything, we may assume the weak limit $x = 0$. By Chapter 5 #47 (Assignment 3), $\{x_n\}$ is bounded, say $\|x_n\| \leq M < \infty$ for all $n$. For any subsequence $\{x_{n_k}\}$ we have

$$\left\| \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \right\|^2 = \left\langle \frac{1}{N} \sum_{k=1}^{N} x_{n_k}, \frac{1}{N} \sum_{k=1}^{N} x_{n_k} \right\rangle = \frac{1}{N^2} \sum_{k=1}^{N} \sum_{j=1}^{N} \langle x_{n_j}, x_{n_j} \rangle. \tag{3}$$

It follows that the imaginary part of the last sum on the right side is 0. Thus it is enough to show that $\{x_{n_k}\}$ can be chosen so that the real part of the last sum converges to $\|x\|^2$ as $N \to \infty$. Let $a_{mn} = \mathrm{Re}\langle x_m, x_n\rangle$, so the matrix is symmetric: $a_{mn} = a_{nm}$. Then since $x_n \to 0$ weakly, we have

$$a_{mn} \to 0 \text{ as } n \to \infty \text{ for each } m. \tag{4}$$

Let $n_1 = 1$ and, using (4), choose $n_2$ large enough so $|a_{n_1n_2}| < \frac{1}{2}$ . Then again using (4), choose $n_3 > n_2$ large enough so $|a_{n_1n_3}| < \frac{1}{3}$ and $|a_{n_2n_3}| < \frac{1}{3}$. Continue this way, choosing $n_k > n_{k-1}$ so that $|a_{n_jn_k}| < \frac{1}{k}$ for all $j < k$. Since the matrix is symmetric we also have $|a_{n_kn_j}| < \frac{1}{k}$ for all $k > j$, while for $j = k$ we have $|a_{n_jn_k}| = \|x_{n_j}\|^2 \le M^2$. Therefore

$$\begin{aligned}\left|\frac{1}{N^2}\sum_{k=1}^{N}\sum_{j=1}^{N}\langle x_{n_j}, x_{n_k}\rangle\right| &\le \frac{1}{N}M^2 + \frac{2}{N^2}\sum_{1\le j<k\le N}|a_{n_j}a_{n_k}| \\ &\le \frac{1}{N}M^2 + \frac{2}{N^2}\left(\frac{1}{2} + 2\cdot\frac{1}{3} + 3\cdot\frac{1}{4} + \cdots + (N-1)\cdot\frac{1}{N}\right) \\ &\le \frac{1}{N}M^2 + \frac{2}{N} \\ &\to 0 \text{ as } N \to \infty.\end{aligned}$$

(3)(Ch. 5 #37) We have the isometry $y \mapsto \hat{y}$ from $Y$ to $Y^{**}$. Let $\mathcal{B}_X$ be the unit ball of $X$. The assumption of the problem says that

$$\sup_{x\in\mathcal{B}_X}|\widehat{Tx}(f)| = \sup_{x\in\mathcal{B}_X}|(f\circ T)(x)| < \infty \quad \text{for all } f \in Y^*.$$

Applying Theorem 5.13b to the family $\mathcal{A} = \{\widehat{Tx} : x \in \mathcal{B}_X\}$ we obtain using Theorem 5.8d that

$$\sup_{x\in\mathcal{B}_X}\|Tx\| = \sup_{x\in\mathcal{B}_X}\|\widehat{Tx}\| < \infty.$$

This says that $T$ is bounded.

(4)(Ch. 6 #5) Let $0 < p < q < \infty$. Suppose $X$ contains sets of arbitrarily small positive measure. Inductively we can choose $F_1, F_2, \ldots$ with $0 < \mu(F_{n+1}) < \frac{1}{3}\mu(F_n)$ for all $n \ge 1$. Then $\mu(F_{n+j}) < 3^{-j}\mu(F_n)$ for all $j \ge 1$. Let $E_n = F_n\backslash(\cup_{k>n}F_k)$, so $E_1, E_2, \ldots$ are disjoint with

$$3^{-n+1}\mu(E_1) \ge \mu(E_n) \ge \mu(F_n) - \sum_{k>n}\mu(F_k) \ge (1 - \sum_{j\ge1}3^{-j})\mu(F_n) \ge \frac{1}{2}\mu(F_n) > 0.$$

Let $a_n = \mu(E_n)^{-1/q}$ and $f = \sum_n a_n \chi_{E_n}$. Then $\|f\|_q^q = \sum_n a_n^q \mu(E_n) = \sum_n 1 = \infty$ so $f \notin L^q$. But

$$\begin{aligned}\|f\|_p^p = \sum_n a_n^p \mu(E_n) &= \sum_n \mu(E_n)^{(q-p)/q} \\ &\leq \sum_n (3^{-n+1}\mu(E_1))^{(q-p)/q} \\ &\leq (\text{const.}) \sum_n (3^{-(q-p)/q})^n \\ &< \infty,\end{aligned}$$

since $3^{-(q-p)/q} < 1$.

Conversely suppose $X$ does not contain sets of arbitrarily small positive measure, and $f \in L^p$. Then $\mu(\{x : |f(x)| > t\}) \to 0$ as $t \to \infty$, so this measure must actually equal 0 for $t$ large. This means $\|f\|_\infty < \infty$. Now for $0 \leq s < \|f\|_\infty$ we have $s^q \leq \|f\|_\infty^{q-p} s^p$, so we have $|f(x)|^q \leq \|f\|_\infty^{q-p} |f(x)|^p$ for almost every $x$. Therefore $f \in L^q$. Thus $L^p \subset L^q$.

(5) FIRST SOLUTION, USING THE FULL HINT: Since $X$ is infinite dimensional, we can choose $x_1$ with $\|x_1\| = 1$ and then inductively for each $n$ choose $x_{n+1} \notin \text{span}(x_1, \ldots, x_n)$ with $\|x_{n+1}\| = 1$. Let $S_n = \text{span}(x_1, \ldots, x_n)$ and $S = \cup_n S_n$; then $S$ is a subspace. Since $S_n$ is finite dimensional it is closed (Exercise 18 Chapter 5, done in 525a); therefore $1 \geq d(x_{n+1}, S_n) > 0$. We claim $S$ is not closed. If $a_n \in \mathbb{C}$ with $\sum_n |a_n| < \infty$ then $\sum_n a_n x_n$ converges absolutely, hence it converges to some $x$, and the sum $x = \lim_n \sum_{k=1}^n a_k x_k$ is a limit point of $S$. We claim we can choose the $a_n$ so that $a_n > 0$ and

$$\sum_{k \geq n+2} a_k < d(a_{n+1}x_{n+1}, S_n) \text{ for all } n. \tag{5}$$

If we can do this, then using the fact that $d(u + v, S_n) \leq d(u, S_n) + \|v\|$ we get

$$d\left(\sum_{k \geq n+1} a_k x_k, S_n\right) \geq d(a_{n+1}x_{n+1}, S_n) - \left\|\sum_{k \geq n+2} a_k x_k\right\| \geq d(a_{n+1}x_{n+1}, S_n) - \sum_{k \geq n+2} |a_k| > 0,$$

so $\sum_{k \geq n+1} a_k x_k \notin S_n$, so the limit point $x = \sum_{k \leq n} a_k x_k + \sum_{k \geq n+1} a_k x_k \notin S_n$, for all $n$. Thus $x \notin S$, which shows that $S$ is not closed.

For inequality (5) it is sufficient that $a_k < 2^{-(k-n-1)} d(a_{n+1}x_{n+1}, S_n)$ for all $k \geq n + 2$ and all $n$. Thus we can choose $a_1 = a_2 = 1$ and then inductively $a_k < \min_{2 \leq j < k} 2^{-(k-j)} d(a_j x_j, S_{j-1})$.

SECOND SOLUTION, NOT USING THE FULL HINT: Let $S_n, S$ be as in the first solution. Since $S_n$ is finite dimensional it is closed (Exercise 18 Chapter 5, done in 525a.) If

$x \in S_n$ and $U$ is a neighborhood of $x$ in $X$, then $x + \epsilon x_{n+1} \in U \backslash S_n$ for all sufficiently small $\epsilon$, so $U \not\subset S_n$. Thus $S_n$ contains no open set, so $S_n$ is nowhere dense. It follows from the Baire Category Theorem that $S$ is not complete. Since $X$ is Banach, this means $S$ is not closed.

(6)(a) At each $x$ the intervals $[x, x + \frac{1}{n}), n \geq 1$, form a neighborhood base, so $(\mathbb{R}, \mathcal{T}_h)$ is first-countable. For each $x$, $[x, x+1)$ is an open neighborhood of $x$, so if $\mathcal{B}$ is a base then $\mathcal{B}$ includes an open neighborhood $U_x$ of $x$ contained in $[x, x+1)$. This means the smallest number in $U_x$ is $x$, so we cannot have $U_x = U_y$ for $x \neq y$. Thus $\{U_x : x \in \mathbb{R}\}$ is uncountable, meaning $\mathcal{B}$ is uncountable. Thus $(\mathbb{R}, \mathcal{T}_h)$ is not second countable.

(b) Let $K$ be compact in $(\mathbb{R}, \mathcal{T}_h)$ and let $x \in K$. For any sequence which converges in $(\mathbb{R}, \mathcal{T}_h)$, say $z_n \to z$, we must have $z_n \geq z$ eventually, since sets $[z, z+a)$ form a neighborhood base. If there exists a sequence $y_n \nearrow x$ (in the usual topology) in $K$, this means no subsequence $y_{n_k}$ converges in $(\mathbb{R}, \mathcal{T}_h)$, contradicting compactness of $K$, by Theorem 4.29. Thus there is no such $\{y_n\}$, meaning that there is an interval $(x - \delta_x, x)$ which does not intersect $K$, and this is true for each $x \in K$. These intervals must be disjoint for distinct $x \in K$, and every collection of disjoint open intervals in $\mathbb{R}$ is countable since each interval contains a rational number. Thus $K$ is countable.

(c) Take any decreasing converging sequence, say $\{\frac{1}{n}\}$, and let $K = \{\frac{1}{n}, n \geq 1\} \cup \{0\}$. If $\mathcal{G}$ is an open cover of $K$ then there exists $U \in \mathcal{G}$ with $0 \in U$, so there exists an $h$-interval $[0, a) \subset U$. Then $1/n \in U$ for all $n > 1/a$, and for each $j \leq 1/a$ there exists $U_j \in \mathcal{G}$ with $1/j \in U_j$. Thus $\{U\} \cup \{U_j, j \leq 1/a\}$ is a finite subcover, which shows $K$ is compact.