sparse recovery of positive signals with minimal expansion

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joint work with

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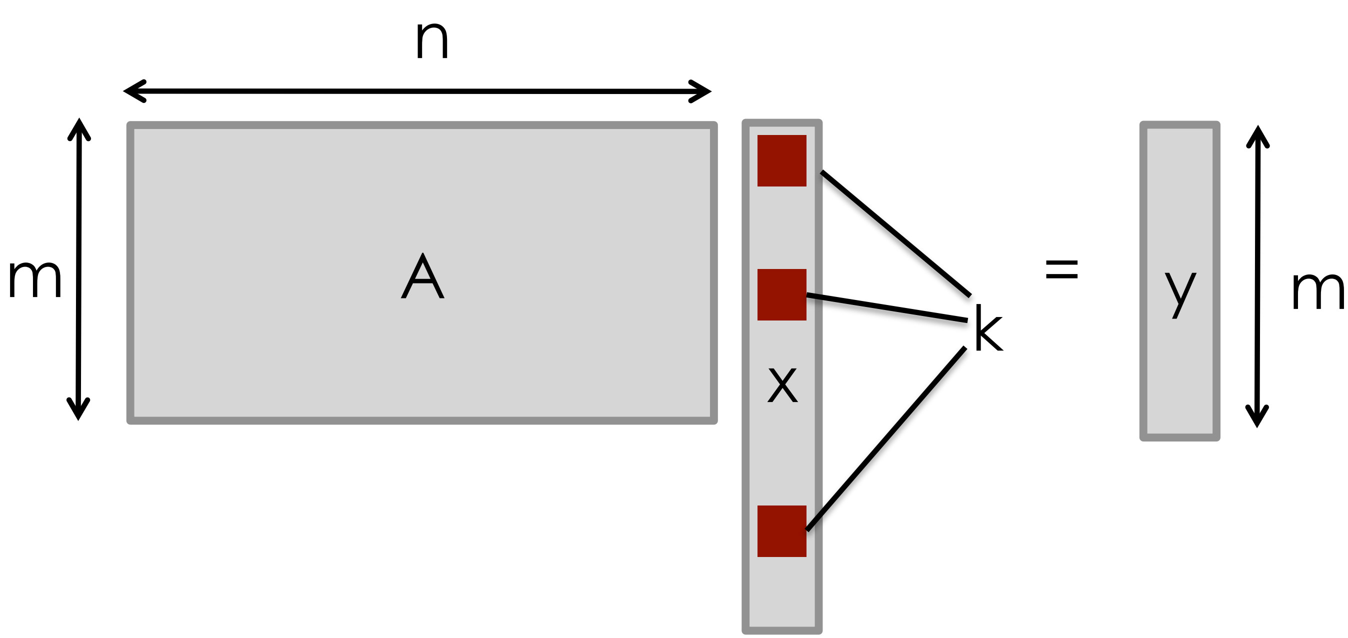
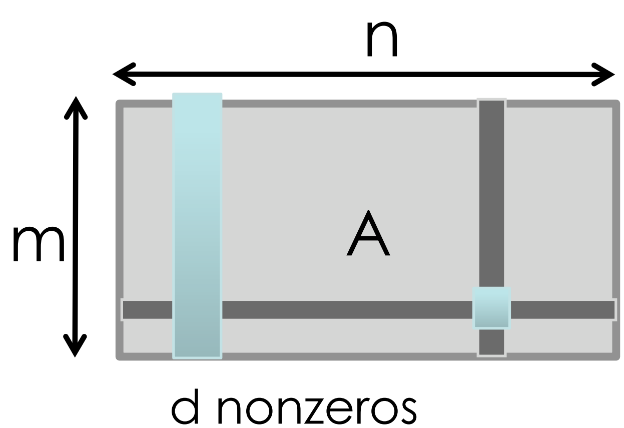
exact sparse recovery

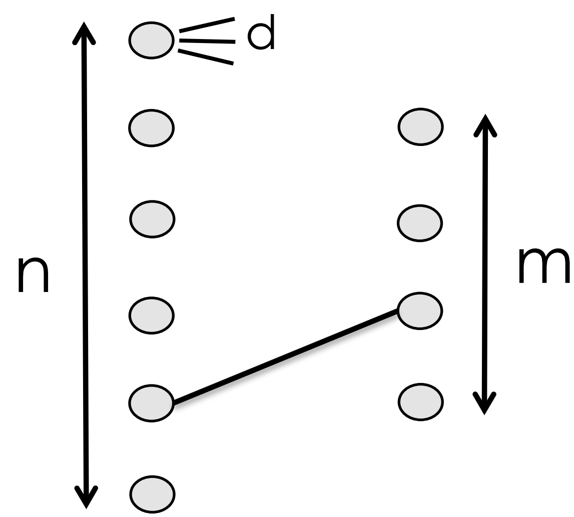


$$\min ||\mathbf{x}||_0 \text{ s.t } \mathbf{Ax} = \mathbf{y} \quad \overset{?}{\Leftrightarrow} \quad \min ||\mathbf{x}||_1 \text{ s.t } \mathbf{Ax} = \mathbf{y}$$

A must correspond to an expander

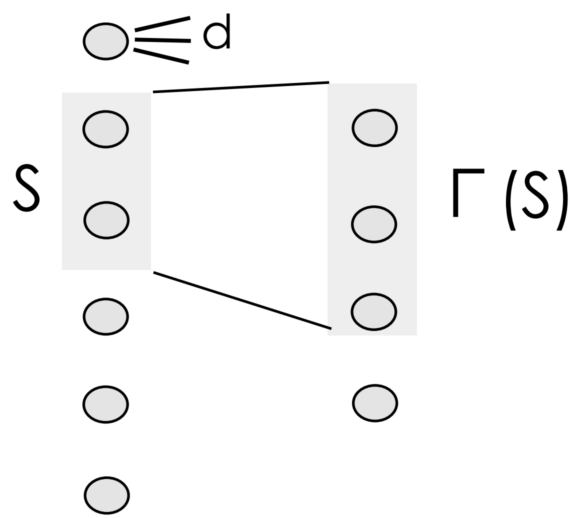
from matrices to bipartite graphs





(α, γ)-expander graphs

an n,m bipartite graph with left degree d is an (α, γ) –expander if:

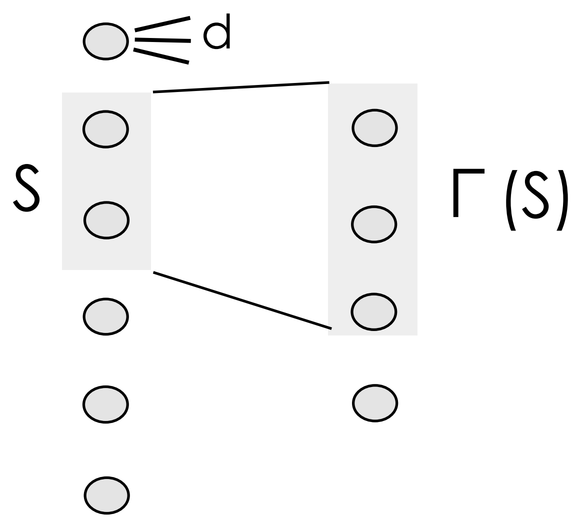


for every set S,

$\Gamma(S) = d|S|$

(α, γ)-expander graphs

an n,m bipartite graph with left degree d is an (α, γ) –expander if:



$\gamma \geq 1/2$, unique neighbor
Yields RIP-1 matrices (GIKS'08)

$\gamma \geq 3/4$, bit flipping
(Sipser et al. Xu et al.)

$\gamma = 1/d$ (critical expansion)

for every set S, such that $|S| \leq \alpha n$,

$$\Gamma(S) \geq \gamma d |S|$$

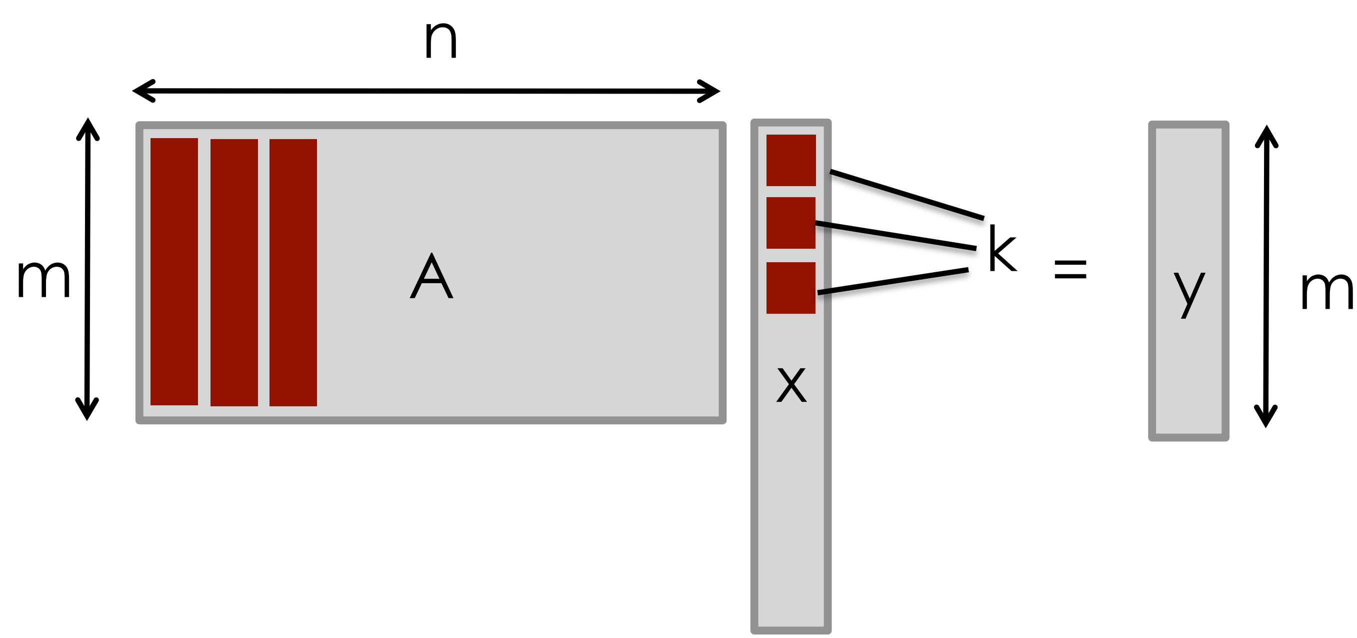
who cares?

- required expansion influences the size of recoverable signals.
- set $m=n/2$ (rate 1/2 code). what is the recoverable fraction $k= cn$
- for $\gamma \geq 1/2$, $k=$**0.000001** n
- for $d=9$, $\gamma \geq 1/9$, yields $k=$ **0.05** n

critical expansion is necessary

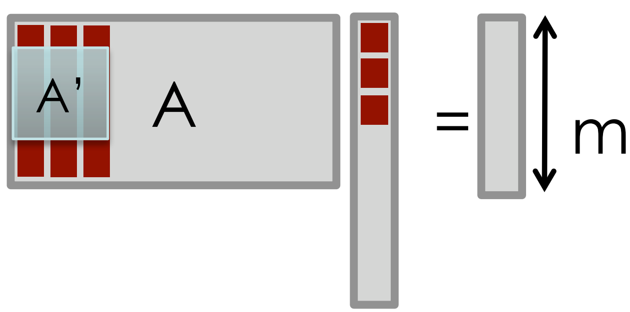
lemma: every matrix that works for compressive sensing of k-sparse signals must be an (α=k/n ,γ=1/d)-expander.

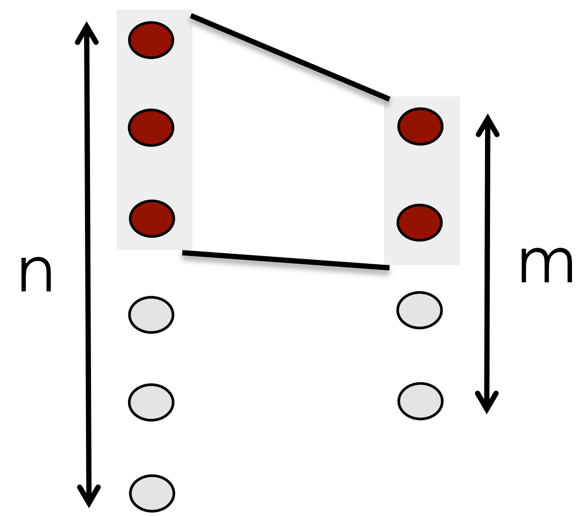
i.e. for every set S, s.t. $|S| \leq k$, $\Gamma(S) \geq |S|$



critical expansion: proof

assume otherwise: there exists a set of size k that contracts.



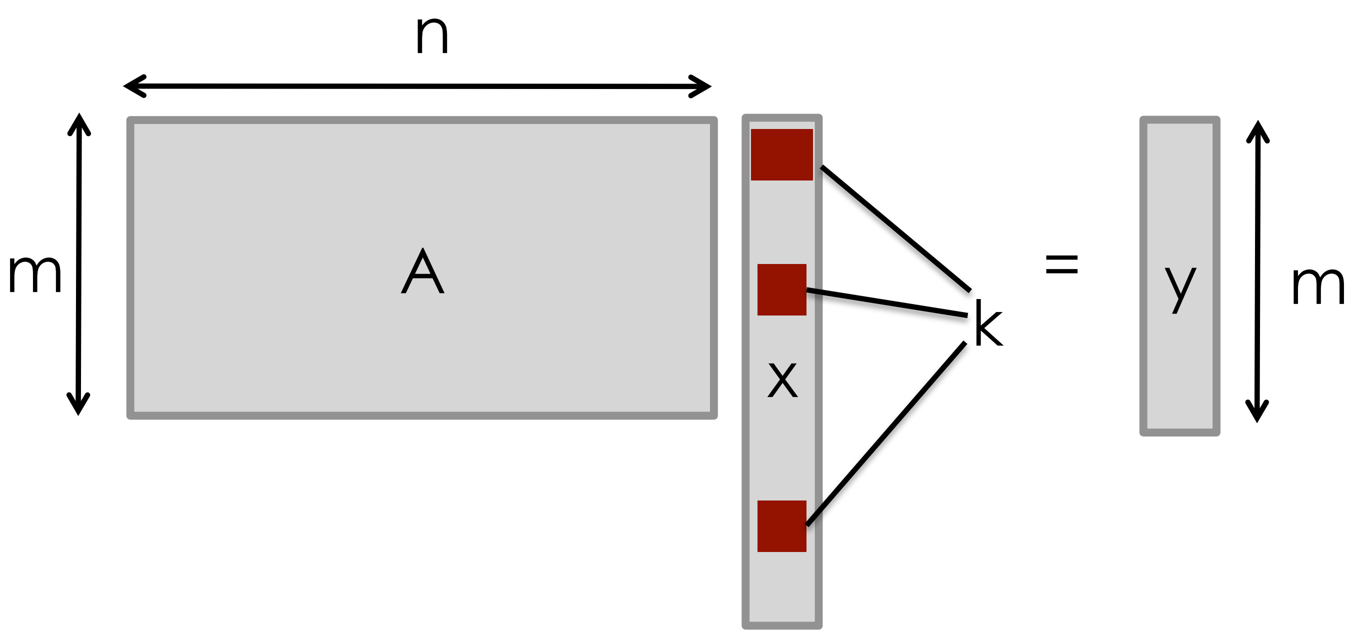


contraction means that there is no perfect matching, so A' submatrix must have rank less than k.

ok, critical expansion is necessary. Is it sufficient?

yes, for positive signals.
unknown in general.

sparse recovery for positive signals



$$\min ||\mathbf{x}||_0 \text{ s.t } \mathbf{Ax} = \mathbf{y} \quad x \geq 0 \quad ? \quad \min ||\mathbf{x}||_1 \text{ s.t } \mathbf{Ax} = \mathbf{y} \quad x \geq 0$$

(Donoho et al. Bruckstein et al. Dai et al. all for dense A)

critical expansion is sufficient (at least for positive signals)

theorem: from any (α,γ=1/d)-expander we can construct a measurement matrix A that recovers **all** k=f(α)-sparse positive signals.

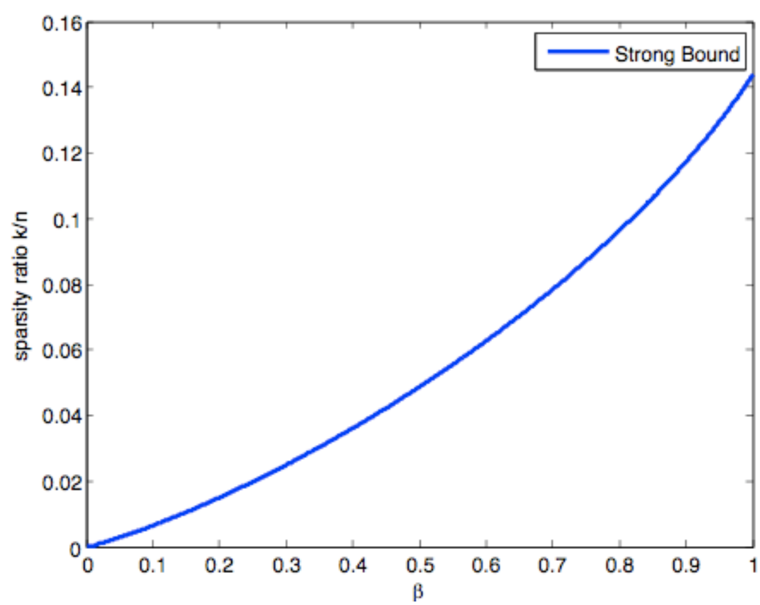
Construct (0,1) matrix from expander.
Perturb non-zero entries (but maintain column sum).

Suffices to construct large 1/d expanders.

first moment method

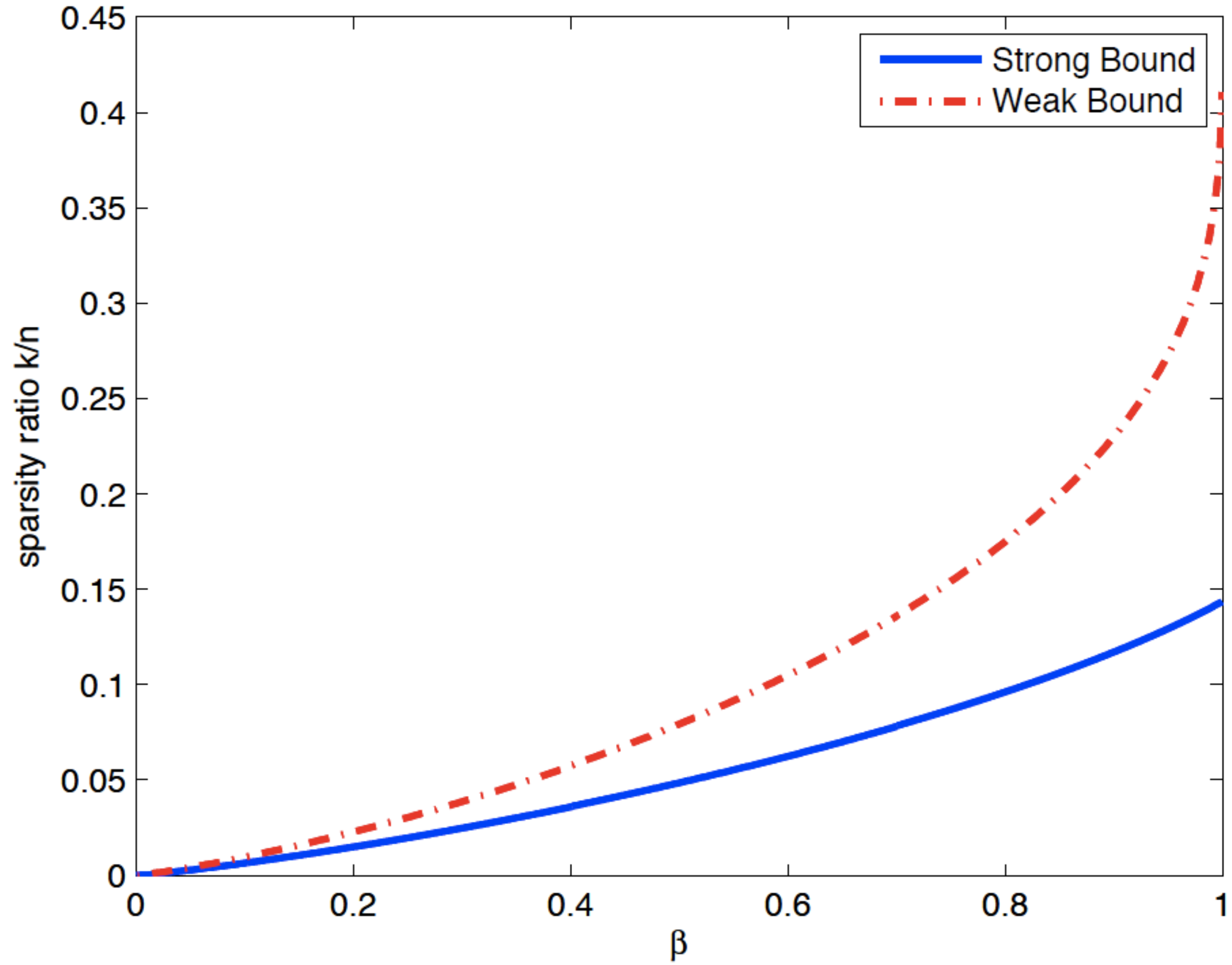
For sufficiently large n, with $m = \beta n$ and $r_0 = \mu n$, there exists a bipartite graph with left vertex size n and right size m which is a $(r_0, \frac{1}{d})$ expander, if

$$d > \frac{H(\mu)+\beta H(\frac{\mu}{\beta})}{\mu \log(\frac{\beta}{\mu})}.$$



Recoverable fraction

worst vs average case for rate β =m/n



Techniques

For the general cs problem, null space characterization is necessary and sufficient (Xu et al. also Donoho et al.)

$$\min ||\mathbf{x}||_0 \text{ s.t } \mathbf{Ax} = \mathbf{y} \Leftrightarrow \min ||\mathbf{x}||_1 \text{ s.t } \mathbf{Ax} = \mathbf{y}$$

iff for every **w**, s.t. **Aw=0**,

$$\forall S \in \{1, 2, \ldots, n\}, |S| = k$$

$$\sum_{i \in S} |\mathbf{w}_i| \leq \sum_{i \in S^c} |\mathbf{w}_i|$$

for every w, there is no small set that contains the majority of the l1 mass

null space condition for positive signals

lemma: for positive sparse signals:

$$\min ||\mathbf{x}||_0 \text{ s.t } \mathbf{A}\mathbf{x} = \mathbf{y}, \; x \geq 0 \iff \min ||\mathbf{x}||_1 \text{ s.t } \mathbf{A}\mathbf{x} = \mathbf{y}, \; x \geq 0$$

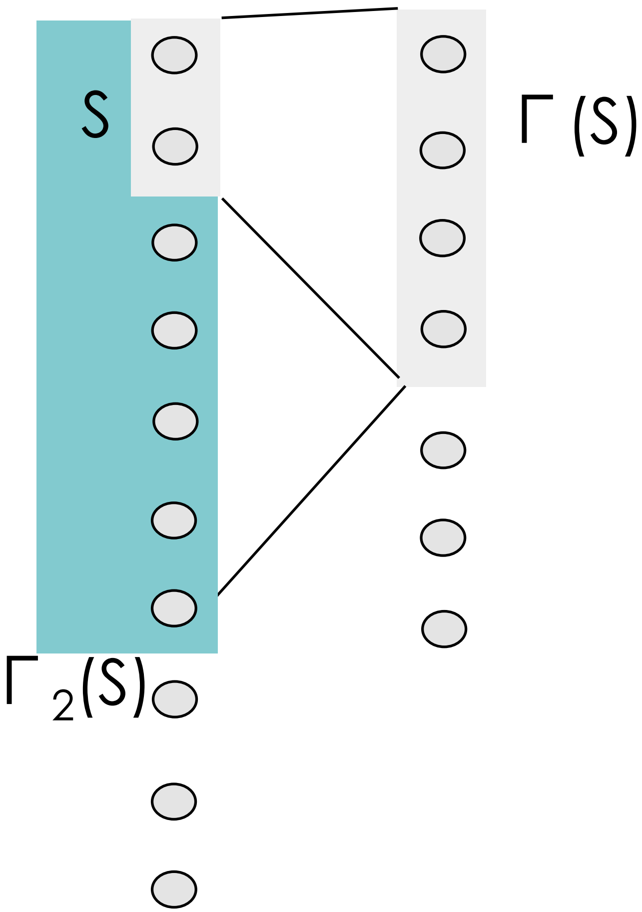
iff

For d-regular bipartite graphs:

$\forall \mathbf{w} \in \mathcal{N}(\mathbf{A})$, $\mathbf{w}$ has at least k negative elements.

every w has large negative support

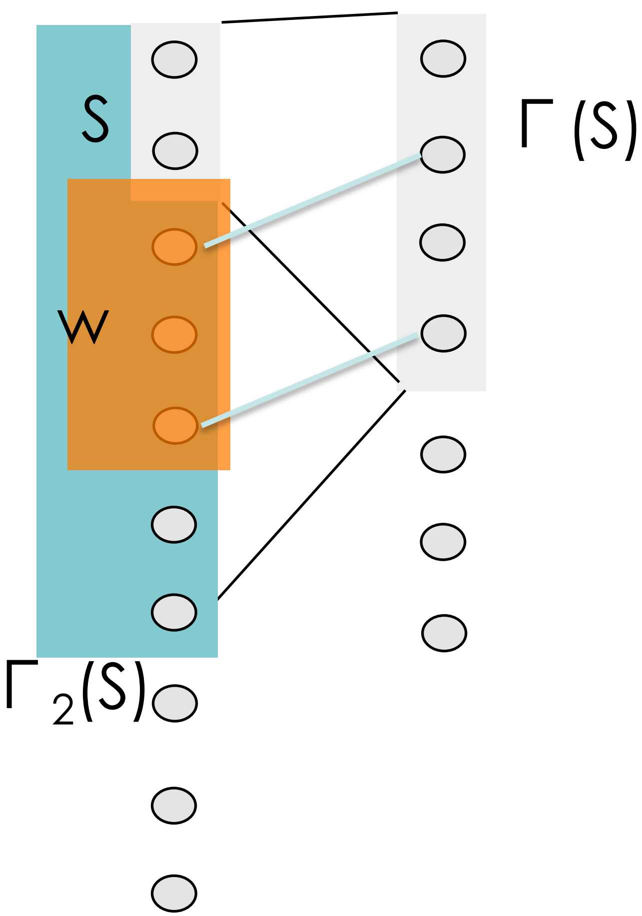
combinatorial condition for positive signals



lemma: A specific set S can be recovered if:

Every subset of $\Gamma_2(S)$, of size up to $\Gamma(S)+1$ has expansion 1/d

combinatorial condition for nullspace



•expansion 1/d ⇔ perfect matching (Hall's theorem)

•matching+ perturbations => full rank submatrices

•assume there existed some w, such that A'w=0.

•if A' is full rank, w must be 0.

•so w cannot be adjacent to a perfect matching.

fin