Connectivity and Equilibrium in Random Games

Constantinos Daskalakis[*] Alexandros G. Dimakis[†] Elchanan Mossel[‡]

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Abstract

We study *how the structure of the interaction graph* of a game affects the existence of pure Nash equilibria. In particular, for a fixed interaction graph, we are interested in whether there exist pure Nash equilibria which arise when random utility tables are assigned to the players. We provide conditions for the structure of the graph under which equilibria are likely to exist and complementary conditions which make the existence of equilibria highly unlikely. Our results have immediate implications for many deterministic graphs and generalize known results for random games on the complete graph. In particular, our results imply that the probability that bounded degree graphs have pure Nash equilibria is exponentially small in the size of the graph and yield a simple algorithm that finds small non-existence certificates for a large family of graphs. We then show that as $n \to \infty$, any graph on n vertices with expansion $(1 + \Omega(1)) \log n$ will have the number of equilbria distributed as a Poisson random variable with parameter 1.

In order to obtain a refined characterization of the degree of connectivity associated with the existence of equilibria, we study the model in the random graph setting. In particular, we look at the case where the interaction graph is drawn from the Erdős-Rényi, $G(n, p)$, where each edge is present independently with probability p. For this model we establish a *double phase transition* for the existence of pure Nash equilibria as a function of the average degree pn consistent with the non-monotone behavior of the model. We show that when the average degree satisfies $np > (2 + \Omega(1)) \log n$, the number of pure Nash equilibria follows a Poisson distribution with parameter 1. When $1/n << np < (0.5 - \Omega(1)) \log n$ pure Nash equilibria fail to exist with high probability. Finally, when $np << 1/n$ a pure Nash equilibrium exists with high probability.

1 Introduction

In recent years, there has been a convergence of ideas coming from computer science, social sciences and economic sciences as researchers in these fields attempt to model and analyze the characteristics and dynamics of large complex networks, such as the web graph, social networks and recommendation networks. From the computational perspective, it has been recognized that the successful design of algorithms performed on such networks, including routing, ranking and recommendation algorithms, must take into account the social dynamics and economic incentives as well as the technical properties that govern network growth [25, 28, 19].

[*]EECS, UC Berkeley, email: costis@eecs.berkeley.edu. Supported by NSF award CCF-0635319, NSF awards DMS 0528488 and DMS 0548249 (CAREER), a research gift from Yahoo! Research and a MICRO grant.

[†]EECS, UC Berkeley, email: adim@eecs.berkeley.edu. Supported by NSF awards CCR-0219722 and CCR-0330514.

[‡]Statistics, UC Berkeley, email: mossel@stat.berkeley.edu. Supported by a Sloan fellowship in Mathematics, NSF awards DMS 0528488 and DMS 0548249 (CAREER) and by ONR grant N0014-07-1-05-06.

Game theory has been very successful in modeling strategic behavior in large networks of economically incentivized entities. In the context of routing, for instance, it has been employed to study the effect of selfishness on the efficiency of a network, whereby the performance of the network at equilibrium is compared to the case where a central authority can simply dictate a solution [30, 32, 33, 7]. The effect of selfishness has been studied in several other settings, see e.g. load balancing [8, 9, 20, 31], facility location [35], network design[4].

One simple way to model interactions between agents in a large network is via the notion of a *graphical game* [18]: a graph $G = (V, E)$ is defined whose vertices represent the players of the game and an edge $(v, w) \in E$ corresponds to strategic interaction between the players v and w; each player $v \in V$ has a finite set of strategies S_v and an *utility, or payoff, table* u_v which assigns a real number $u_v(\sigma_v, \sigma_{\mathcal{N}(v)})$ to every selection of strategies for player v and the players in v's neighborhood, denoted by $\mathcal{N}(v)$. A *pure Nash equilibrium* (PNE) of the game is some state or strategy profile σ of the game which assigns to every player v a single strategy $\sigma_v \in S_v$ in such a way that no player has a unilateral incentive to deviate. Equivalently, for every player $v \in V$,

$$u_v(\sigma_v, \sigma_{\mathcal{N}(v)}) \geq u_v(\sigma'_v, \sigma_{\mathcal{N}(v)}), \text{ for every strategy } \sigma'_v \in S_v. \tag{1}$$

When condition (1) is satisfied, we say that the strategy σ_v is a *best response to* $\sigma_{\mathcal{N}(v)}$. The concept of a pure strategy Nash equilibrium is more compelling, decision theoretically, than the concept of a mixed strategy equilibrium and it is therefore interesting to study how the number of PNE depends on the interaction structure of a game.

Graphical games provide a more compact model of large networks of interacting agents, than normal form games, in which the game is described as if it were played on the complete graph. Besides the compact description, one of the motivations for the introduction of graphical games is their intuitive affinity with graphical statistical models; indeed, several algorithms for graphical games do have the flavor of algorithms for solving Bayes nets or constraint satisfaction problems [21, 24, 15, 12, 10].

In the other direction, the notion of a pure Nash equilibrium provides a *new species of constraint satisfaction problems*. Notably one in which, for any assignment of strategies (values) to the neighborhood of a player (variable), there is always a strategy (value) for that player which makes the constraint (1) corresponding to that player satisfied (i.e. being in best response). The reason why it might be hard to satisfy simultaneously the constraints corresponding to all players is the long range correlations that may appear between players. Indeed, deciding whether a pure Nash equilibrium exists is NP-hard even for very sparse graphical games [15].

Viewed as a constraint satisfaction problem, the problem of existence of Nash equilibria poses interesting challenges. First, it is easy to see that for natural models as the one described here, the *expected number* of Nash equilibria is 1 for *any graph*, while for most other constraint satisfaction problems, the expected number of solution is exponential in the size of the graph with different exponents corresponding to different density parameters. Second, unlike most constraint satisfaction problems studied before, the problem of existence of pure Nash equilibria is not a-priori monotone in any sense. It is remarkable that given these novel features of the problem it is possible to obtain a result establishing a double phase transition as described below.

1.1 Outline of Main Results

We obtain two types of results. In the first type we consider the existence of pure Nash equilibria (PNE) on random graphs and obtain exact information on the probability of existence of PNE in terms of the density of the random graphs. The second type of results concerns general graphs where we obtain conditions under which PNE do not exist with high probability and also propose an efficient algorithm for finding witnesses of non existence of PNE.

We start be defining the notion of a random game.

Definition 1.1 *We study the number of PNE for graphical games when each player is assigned a payoff table whose entries are independently chosen from an atom-less distribution. In this case the existence of PNE is only determined by the* best response tables. *The best response table for player v is a binary table with the same dimensions as the payoff table; the entry indexed by $(\sigma_v, \sigma_{\mathcal{N}(v)})$ is set to 1 if σ_v is a best response to $\sigma_{\mathcal{N}(v)}$, otherwise it is set to 0. It is thus suffices to consider the uniform measure over best response tables which have exactly one value equal to 1 in the row indexed by $\sigma_{\mathcal{N}(v)}$. We will further assume that each player can take only two actions.*

The problem of determining the number of PNE for random games with independent payoffs has been studied extensively: The early work of Goldberg et al. [14] computes the probability that a two-person random game with independent payoff tables has at least one PNE. Subsequently, Dresher [11] and Papavassilopoulos [26] generalized this result for an n-player game on the complete graph. Powers [27] and Stanford [34] show that the number of PNE converges to a Poisson(1) random variable, for the complete graph as the number of players increases. Rinott et al. [29] investigate the asymptotic distribution of PNE when there are positive or negative dependencies among the payoff tables. In this paper we investigate the number of PNE on random games with independent payoffs, for various deterministic and random graphs, generalizing the existing results beyond games on the complete graph. The main results discussed in this paper have been presented in a number of talks [23, 22] which resulted in a few manuscripts studying random games on various graph topologies.

1.1.1 PNE on Random Graphs

The first question we address is *what is the average degree required for a game to have pure Nash equilibria?* To study this question it is natural to consider families of graphs with different densities and see how the probability of PNE correlates with the density of the graph. We do so by considering graphs drawn from the Erdős-Rényi, $G(n,p)$, model where each edge is present independently with probability p. Interestingly, the existence of a PNE is *not monotone* in p: an empty graph trivially has a PNE, a complete graph has a PNE with asymptotic probability $1-\frac{1}{e}$ (see [11, 29]), but our results indicate that when p is in some intermediate regime, a PNE does not exist w.h.p. as n increases. Surprisingly, the Poisson convergence result for games on the complete graph [27, 34] generalizes to random games on random graphs as long as the degree are at least *logarithmic* in the number of nodes in the graph. We also show that if the sparsity further increases, PNE do not exist with high probability until the graph becomes essentially empty, in which case PNE appear again with probability 1.

Our study here is an example of studying satisfiability of general constraint satisfaction problems. The question is to investigate the effect of the structure of the constraint graph on the satisfiability of the problems defined on the graph, as well as the computational complexity required to solve them. In the context of SAT formulas the key parameter is the density of the hypergraph defined on the variables, with a hyperedge corresponding to each clause, see e.g., [13, 1]. In other settings, other structural properties are important, for instance measures of the cyclicity of the graph [6, 16].

Traditionally, in order to prove that a solution does not exist one either uses the first moment method [3] or finds a witness of unsatisfiability. The second moment method has been often used to show the existence of satisfying assignment and its refinements provide some of the best bounds for satisfiability to date [2].

As noted before, in our case the expected number of satisfying assignments is 1 for any graph. This suggests that the analysis of the problem should be extremely hard. Instead, we show how the second moment method can be used to show the existence of PNE in random games with sufficiently large density, and further use Stein's [5] method to prove that in this case for almost all graphs the distribution of the number of PNE is asymptotically Poisson. Our proof that at lower densities there are no PNE uses small witnesses. More formally, we establish a *phase transition* described by the following theorems:

Theorem 1.2 (High connectivity) *Let Z denote the number of PNE in a random $G(n,p)$-game where $p = \frac{(2+\epsilon)\log n}{n}$ and $\epsilon = \epsilon(n) > 0$. Then the distribution of Z converges to a Poisson(1) random variable. Further, for any n, with probability at least $1-2n^{-\epsilon/8}$ over the random graph it holds that the total variation distance between the distribution of Z and the distribution of a Poisson(1) r.v. W is bounded by:*

$$||Z - W|| \leq O(n^{-\epsilon/8}) + \exp(-\Omega(n)). \tag{2}$$

(note that the two terms can be of the same order when ϵ is of order $n/\log n$)

Theorem 1.3 (Medium Connectivity) *For $p \leq 1/n$ the probability of PNE is bounded by:*

$$\exp(-\Omega(n^2 p)).$$

For $p(n) = g(n)/n$, where $g(n) > 1$, the probability of PNE is bounded by:

$$\exp(-\Omega(e^{\log n - 2g(n)})).$$

In particular the probability of PNE goes to 0 for

$$\frac{1}{n^2} << p < (0.5 - \epsilon'(n))\frac{\log n}{n}.$$

for $\epsilon'(n) = \frac{\log\log n}{\log n}$.

Theorem 1.4 (Low Connectivity) *For every constant $c > 0$, if $p \leq \frac{c}{n^2}$, the probability of PNE is at least*

$$\left(1 - \frac{c}{n^2}\right)^{\frac{n(n-1)}{2}} \longrightarrow e^{-\frac{c}{2}}$$

Note that our lower and upper bounds for $G(n,p)$ leave a small gap, between $p = \frac{0.5\log n}{n}$ and $p = \frac{2\log n}{n}$. The behavior of the number of PNE in this regime remains open. It is natural to ask weather PNE appear exactly at the point where our witness for nonexistence of PNE becomes unlikely to exist. In other words we believe that our 'indifferent matching pennies' witnesses (defined subsequently) are (similarly to isolated vertices in connectivity) the smallest structures that prevent the existence of pure Nash equilibria and the last ones to disappear.

1.1.2 General Graphs

We provide conditions for existence and non-existence of pure Nash equilibria in games defined on deterministic graphs. The existence of pure Nash equilibria is guaranteed by sufficient *expansion* properties of the underlying graph. The notion of expansion that we shall use is defined next.

Definition 1.5 *A graph $G = (V, E)$ is a* strong (α, δ)-expander *iff every set V'such that $|V'| \leq \delta|V|$ has $|\mathcal{N}(V')| \geq \alpha|V'|$ neighbors and every set V' such that $|V'| > \delta|V|$ has $|\mathcal{N}(V')| = |V|$ neighbors. Here we let*

$$\mathcal{N}(V') = \{w \in V : \exists u \in V' \text{ with } (w, u) \in E\}.$$

(Note in particular that $\mathcal{N}(V')$ may intersect V').

We show the following result.

Theorem 1.6 (Expander Graphs) *Let Z denote the number of PNE in a random graphical game defined on a graph G on n vertices which is a strong (α, δ)-expander, where $\alpha = (1 + \epsilon) \log_2 n$, $\delta = \frac{1}{\alpha}$ and $\epsilon > 0$. Then the distribution is of Z is approximated by a Poisson(1) random variable in the followng sense: The total variation distance between the distribution of Z and the distribution of a Poisson(1) r.v. W is bounded by:*

$$||Z - W|| \leq O(n^{-\epsilon}) + O(2^{-n/2}). \tag{3}$$

We provide next a complementary condition for the non-existence of PNE. We introduce the following structural property: an edge of a graph will be called *d-bounded* if both its vertices have degrees smaller or equal to d. We bound the probability of the existence of a PNE as a function of the number of such edges:

Theorem 1.7 *Consider a random game on a graph G which has m vertex disjoint edges that are d bounded has no PNE with probability at least:*

$$1 - \exp\left(-m\left(\frac{1}{8}\right)^{2^{2d-2}}\right). \tag{4}$$

In particular, if the graph has m edges that are d bounded then the game has no PNE with probability at least:

$$1 - \exp\left(-\frac{m}{2d}\left(\frac{1}{8}\right)^{2^{2d-2}}\right). \tag{5}$$

Moreover, there exists an algorithm of complexity $O(m2^{d+2})$ for proving that PNE does not exist with success probability given by (4) and (5) respectively.

More generally, assign to each edge $(u, v) \in E$ weight $w_{(u,v)} := -\log(1 - p_{(u,v)})$, where $p_{(u,v)} = 8^{-2^{d_u+d_v-2}}$, where d_u, d_v are the degree of u and v respectively. Suppose that $\mathcal{E}$ is a maximal weighted independent edge set with value $w_{\mathcal{E}}$. Then the probability that there exists no PNE is at least

$$1 - \exp\left(-w_{\mathcal{E}}\right).$$

An easy consequence of this result is that many sparse graphs, like the line and the grid, do not have pure Nash equilibria with high probability as the number of players increases.

The proof of the theorem is based on the following witness for the non-existence of PNE: We say that players a and b play the *indifferent matching pennies game* if their payoff tables are defined as follows.

Payoffs to player a :

	b plays 0, any $\sigma_{\mathcal{N}(a)\setminus\{b\}}$	b plays 1, any $\sigma_{\mathcal{N}(a)\setminus\{b\}}$
a plays 0	1	0
a plays 1	0	1

Payoffs to player b :

	a plays 0, any $\sigma_{\mathcal{N}(b)\setminus\{a\}}$	a plays 1, any $\sigma_{\mathcal{N}(b)\setminus\{a\}}$
b plays 0	0	1
b plays 1	1	0

Note that if a graphical game contains an edge (u, v) so that players u and v play the indifferent matching pennies game then the game has no PNE.

The indifferent matching pennies game provides a *small witness* the non-existence of pure Nash equilibria, which is a co-NP complete problem for bounded degree graphical games. Our analysis implies that, with high probability over bounded degree graphical games, there are short proofs for the non-existence of pure Nash equilibria which can be found efficiently. A related analysis and randomized algorithm was introduced for mixed Nash equilibria for 2 player games by Bárány et al. [17].

1.2 Acknowledgement

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2 Random Graphs

2.1 High Connectivity

In this section we establish that for the random graph model when the average degree satisfies $pn > (2 + \epsilon)\log n$ the distribution of PNE converges to a Poisson(1) random variable, with high probability over random graphs. This implies in particular that a PNE exists with probability that converges to $1 - \frac{1}{e}$ as the size of the network $n \to \infty$ and $pn > (2 + \epsilon)\log n$.

As in the proof of [29] for the complete graph we will use the following result by Arratia et al.[5] that is based on Stein's method. It will be useful below to denote by W_λ a Poisson random variable with parameter λ. We will write W for W_1. For two random variables Z, Z' taking values in $0, 1, \ldots$ we define their *total variation distance* $||Z - Z'||$ as:

$$||Z - Z'|| = \frac{1}{2}\sum_{i=0}^{\infty} |Z(i) - Z'(i)|$$

Lemma 2.1 *Consider arbitrary dependent Bernoulli random variables X_i, $i = 0, \ldots, N$. For each i, define a* neighborhood of dependence *B_i of X_i such that B_i satisfies that $(X_j : j \in B_i^c)$ are* independent *of X_i.*

Let

$$Z = \sum_{i=0}^{N} X_i = 1, \quad \lambda = \mathsf{E}[Z]. \tag{6}$$

and

$$b_1 = \sum_{i=0}^{N} \sum_{j \in B_i} \mathbb{P}[X_i = 1]\mathbb{P}[X_j = 1], \tag{7}$$

$$b_2 = \sum_{i=0}^{N} \sum_{j \in B_i \setminus \{i\}} \mathbb{P}[X_i = 1, X_j = 1]. \tag{8}$$

Then if $b_1, b_2 \to 0$ then Z tends to Poisson with mean λ as $n \to \infty$. Further, the total variation distance between the distribution of Z and a Poisson random variable W_λ with mean λ is bounded by

$$||Z - W|| \leq 2(b_1 + b_2). \tag{9}$$

Theorem 1.2: *Let Z denote the number of PNE in a random $G(n,p)$-game where $p = \frac{(2+\epsilon)\log n}{n}$ and $\epsilon = \epsilon(n) > 0$. Then the distribution of Z converges to a Poisson(1) random variable. Further, for any n, with probability at least $1 - n^{-\epsilon/8} - 2^{-n}$ over the random graph it holds that the total variation distance between the distribution of Z and the distribution of a Poisson(1) r.v. W is bounded by:*

$$||Z - W|| \leq O(n^{-\epsilon/8}) + \exp(-\Omega(n)). \tag{10}$$

Proof: For ease of notation, in the sequel we identify pure strategy profiles with integers in $\{0, \ldots, 2^n - 1\}$ mapping an integer to each strategy profile. The mapping is defined so that if the binary expansion of i is $i(1) \ldots i(n)$ then player k plays $i(k)$.

Next, to every strategy profile $i \in \{0, \ldots, N\}$, where $N = 2^n - 1$, we assign an indicator random variable X_i which is 1 if strategy profile i is a PNE. Then the counting random variable

$$Z = \sum_{i=0}^{N} X_i \tag{11}$$

corresponds to the number of pure Nash equilibria, so that the existence of a pure Nash equilibrium is equivalent to the random variable Z being positive. Clearly we have $\mathsf{E}[X_i] = 2^{-n}$ for all i so $\mathsf{E}[Z] = 1$.

There are two sources of randomness the model: the selection of a random graph which determines the interactions of players and the random entries of the payoff tables of the players which are selected independently. An interesting feature of our model is that the neighborhood of dependence B_i of a strategy profile i is a *random set of strategies* that depends on the graph realization (but not the payoff tables). We denote $\mathbb{P}_{(n,p)}[.]$ for probabilities of events over both graphs and payoff tables and let $\mathbb{P}_{\mathcal{G}}[.]$ and $\mathbb{P}_T$ be the (random) measures over graphs and tables respectively. Observe that the tables can only be realized for a given graph (since their dimensions depend on the graph) and therefore $\mathbb{P}_T$ will always be a function of the graph G. The proof relies on identifying B_i for any fixed graph G.

Lemma 2.2

$$B_0 = \{j : \exists k, \forall k' \text{ s.t. } (k,k') \in E(G) \text{ it holds that; } j(k') = 0\}$$

and

$$B_i = i \oplus B_0 = \{i \oplus j : j \in B_0\}$$

where $i \oplus j = (i(1) \oplus j(1), \ldots, i(n) \oplus j(n))$ *and* $\oplus$ *is the exclusive or operation.*

Proof: By symmetry, it is enough to show that X_0 is independent from $\{X_i\}_{i \notin B_0}$. Observe that, in every strategy profile $i \notin B_0$, every player k of the game has at least one neighbor k' playing strategy 1. By independence of the payoff entries, it follows then that whether strategy 0 is a best response for player k in strategy profile 0 is independent of whether strategy $i(k)$ is a best response for player k in strategy profile i, since these events depend on different rows of the payoff matrix of player k indexed by the strategies of the neighbors, which include player k'. ■

Now for any fixed graph G, $b_1(G)$ and $b_2(G)$ are well defined. We will bound

$$\mathsf{E}_{\mathcal{G}}[b_1(G)] = \mathsf{E}_{\mathcal{G}}\left[\sum_{i=0}^{N}\sum_{j \in B_i} \mathbb{P}_T[X_i = 1]\mathbb{P}_T[X_j = 1]\right] = \mathsf{E}_{\mathcal{G}}\left[\frac{1}{(N+1)^2}\sum_{i=0}^{N}|B_i|\right] = \frac{\mathsf{E}_{\mathcal{G}}[|B_0|]}{N+1}. \tag{12}$$

$$\mathsf{E}_{\mathcal{G}}[b_2(G)] = \mathsf{E}_{\mathcal{G}}\left[\sum_{i=0}^{N}\sum_{j \in B_i \setminus \{i\}} \mathbb{P}_T[X_i = 1, X_j = 1]\right] = (N+1)\sum_{j \neq 0} \mathsf{E}_{\mathcal{G}}\left[\mathbb{P}_T[X_0 = 1, X_j = 1]\mathbb{I}[j \in B_0]\right]. \tag{13}$$

From the symmetry of the model, it is clear that the last term depends only on the number of 1's in i denoted s below. To simplify notation we will write Y_s for the indicator that a strategy with the first s players play 1 and all others play 0 is Nash. We will write I_s for the indicator that this strategy is in B_0 (note that I_s is a function of the graph only). We now have:

$$\mathsf{E}_{\mathcal{G}}[b_2] = 2^n \sum_{s=1}^{n}\binom{n}{s}\mathsf{E}_{\mathcal{G}}[I_s \mathbb{P}_T[Y_0 = 1, Y_S = 1]]. \tag{14}$$

$$\mathsf{E}_{\mathcal{G}}[b_1] = 2^{-n}\sum_{s=0}^{n}\binom{n}{s}\mathsf{E}_{\mathcal{G}}[I_s]. \tag{15}$$

Lemma 2.3

$$\mathsf{E}_{\mathcal{G}}[b_1] \le R(n,p) := \sum_{s=1}^{n}\binom{n}{s}2^{-n}\min(1, n(1-p)^{s-1}) \tag{16}$$

$$\mathsf{E}_{\mathcal{G}}[b_2] \le S(n,p) := \sum_{s=1}^{n}\binom{n}{s}2^{-n}\left[(1+(1-p)^s)^{n-s} - (1-(1-p)^s)^{n-s}\right]. \tag{17}$$

Proof: We begin by analyzing $\mathsf{E}_{\mathcal{G}}[b_1]$. Clearly it suffices to bound $\mathsf{E}[I_s]$ by $n(1-p)^s$. This follows from the fact that in order for the strategy with the set S of 1 to be not independent of the all 0 strategy, it is needed there there is at least one player who is not connected to the set S. The probability for each player not to be connected to a neighbor in S is either $(1-p)^s$ or $(1-p)^{s-1}$ and is always at most $(1-p)^{s-1}$. Therefore the probability there is at least one player who is not connected to S is at most $n(1-p)^{s-1}$.

We now analyze $\mathsf{E}_{\mathcal{G}}[I_s \mathbb{P}_T[Y_0 = 1, Y_S = 1]]$. Let S be a set of size s where players play 1. Recall that two strategy profiles will be dependent when there exists at least one player who has all adjacent players not changing their strategy. Observe that if such a player $P_1 \in S$ existed then $\mathbb{P}_T[Y_0 = 1, Y_s = 1] = 0$, as it cannot be that both strategies for P_1 are best response.

Therefore the only contribution to $\mathsf{E}_{\mathcal{G}}[I_s \mathbb{P}_T[Y_0 = 1, Y_S = 1]]$ is from the event that each player in S is adjacent to at least one other players in S (in other words, there is an edge connecting them). Note that given the event above, in order for $I_s = 1$ it must be the case that at least one of the players in S^c must be not adjacent to S.

Let $p_s = \mathbb{P}_{\mathcal{G}}[\nexists \text{ isolated node in S subgraph}]$ and t denote the number of players in S^c, not adjacent to S. Since each player outside S is non-adjacent to S with probability $(1-p)^s$, the probability that t players are not adjacent to S is

$$\binom{n-s}{t}[(1-p)^s]^t(1-(1-p)^s)^{n-s-t}.$$

Moreover, conditioned on the event that t players in S^c are not adjacent to S and all other are adjacent to S we have that the probability that $Y_0 = 1$ and $Y_s = 1$ is:

$$\frac{1}{2^t}\frac{1}{4^{n-t}}.$$

Putting these together we obtain:

$$\begin{aligned}
\mathsf{E}_{\mathcal{G}}[I_s \mathbb{P}_T[Y_0 = 1, Y_S = 1]] &= p_s \sum_{t=1}^{n-s} \binom{n-s}{t}[(1-p)^s]^t(1-(1-p)^s)^{n-s-t}\frac{1}{2^t}\frac{1}{4^{n-t}}, &(18)\\
&= \frac{p_s}{4^n}\left((2(1-p)^s + (1-(1-p)^s))^{n-s} - (1-(1-p)^s)^{n-s}\right) &(19)\\
&= \frac{p_s}{4^n}\left((1+(1-p)^s)^{n-s} - (1-(1-p)^s)^{n-s}\right) &(20)
\end{aligned}$$

and therefore

$$\mathsf{E}_{\mathcal{G}}[b_2] = \sum_{s=1}^{n} 2^{-n}\binom{n}{s} p_s \left[(1+(1-p)^s)^{n-s} - (1-(1-p)^s)^{n-s}\right] \le S(n,p). \quad (21)$$

■

In the appendix we show that

Lemma 2.4

$$S(n,p) \le O(n^{-\epsilon/4}) + \exp(-\Omega(n)),$$

and

$$R(n,p) \le O(n^{-\epsilon/4}) + \exp(-\Omega(n)).$$

We therefore conclude that the same holds for $\mathsf{E}_{\mathcal{G}}[b_1]$ and $\mathsf{E}[b_2]$ and therefore with probability at least $1 - n^{-\epsilon/8} - 2^{-n}$ over the graphs we have

$$\max(b_1, b_2) \leq n^{-\epsilon/8} + \exp(-\Omega(n)).$$

Note by Lemma 2.1 conditioned on the event that $\max(b_1, b_2) \leq O(n^{-\epsilon/8}) + \exp(-\Omega(n))$ we have that:

$$||Z - W|| \leq 2(b_1 + b_2) \leq O(n^{-\epsilon/8}) + \exp(-\Omega(n))$$

as needed. ■

2.2 Medium Connectivity

Theorem 1.3: *For $p \leq 1/n$ the probability of PNE is bounded by:*

$$\exp(-\Omega(n^2 p)).$$

For $p(n) = g(n)/n$, where $g(n) > 1$, the probability of PNE is bounded by:

$$\exp(-\Omega(e^{\log n - 2g(n)})).$$

In particular the probability of PNE goes to 0 for

$$1/n^2 << p << \frac{0.5 \log n}{n}.$$

Proof: Recall the *matching pennies game* described by the following payoff matrices of its two players a and b. It is not hard to see that the game does not have a pure Nash equilibrium.

Payoffs to player a :

	b plays 0	b plays 1
a plays 0	1	0
a plays 1	0	1

Payoffs to player b :

	b plays 0	b plays 1
a plays 0	0	1
a plays 1	1	0

Note that if a graphical game contains two players who are connected to each other, are isolated from the other players, and are playing matching pennies against each other then the graphical game will not have a PNE. The existence of such a witness is precisely what we shall use to establish our result. In particular, we will show that with high probability a random game from our ensemble will contain an isolated edge between players playing a matching pennies game.

We will use the following exposure argument. Label the vertices of the graph by the set $[n]$. Let $\Gamma_1 = [n]$ and perform the following while $|\Gamma_i| \geq n/2$:

- Let j be the minimal value such that $j \in \Gamma_i$.

- If j is adjacent to more than one vertex or to none, then remove j let $\Gamma_{i+1} = \Gamma_i$ where j and all vertices adjacent to j are removed. Now go to the next iteration.
- Otherwise, let j' be the unique neighbor of j. If j' has at least one neighbor, then let $\Gamma_{i+1} = \Gamma_i$ where j, j' and all vertices adjacent to j' are removed. Now go to the next iteration.
- Otherwise check if the edge connecting j and j' is matching pennies. If it is, declare *NO NASH*. Let $\Gamma_{i+1} = \Gamma_i$ where j, j' are removed. Now go to the next iteration.

It is clear that at each iteration the probability of finding an isolated edge is at least $0.25np(1-p)^{2n}$ and therefore the probability of a victory at that stage is at least $\frac{1}{8}0.25np(1-p)^{2n} > 0.01np(1-p)^{2n}$.

Note that the number of vertices removed in the first $m = 0.1n/(np+1)$ stages is bounded by $2m + Bin(mn, p)$ random variable. This follows since the number of vertices removed at each iteration is a subset of the vertex examined, one of it's neighbors and all the neighbor neighbors. Thus at each iteration at most

$$2 + Bin(n, p)$$

vertices are removed. Therefore the probability that the number of stages is less than m is

$$\exp(-\Omega(n)).$$

and the overall probability that the game has PNE is bounded by:

$$\exp(-\Omega(n)) + (1 - 0.01np(1-p)^{2n})^m \leq \exp(-\Omega(n)) + \exp(-\Omega(mnp(1-p)^{2n})) \leq \exp(-\Omega(mnp(1-p)^{2n})).$$

Note that for $p \leq 1/n$ the last expression is

$$\exp(-\Omega(n^2 p)),$$

while for $p \geq g(n)/n$ where $g(n) \geq 1$ the last expression is

$$\exp(-\Omega(n(1-p)^{2n})) = \exp(-\Omega(ne^{-2g(n)})) \exp(-\Omega(e^{\log n - 2g(n)})).$$

■

2.3 Low Connectivity

Theorem 1.4: *For every constant $c > 0$, if $p = \frac{c}{n^2}$ then*

$$\mathbb{P}_{(n,p)}[\exists \text{ a PNE}] = \left(1 - \frac{c}{8n^2}\right)^{\frac{n(n-1)}{2}} + o(1) = e^{-\frac{c}{16}} + o(1)$$

3 Deterministic Graphs

3.1 A Sufficient Condition for Existence of Equilibria: Expansion

We provide a proof of Theorem 1.6.

Proof: We slightly change the notation of the proof of Theorem 1.2, and let X_i for $i = 0, 1, \ldots, N-1 = 2^n - 1$, is the indicator random variable of the event that the strategy profile i is a pure Nash equilibrium. Clearly,

$$\mathsf{E}[Z] = \mathsf{E}\left[\sum_{i=0}^{N-1} X_i\right] = 1.$$

As in the proof of Theorem 1.2, to establish our result, it suffices to bound the following quantities.

$$\begin{aligned} b_1(G) &= \sum_{i=0}^{N-1} \sum_{j \in B_i} \mathbb{P}[X_i = 1]\mathbb{P}[X_j = 1], \\ b_2(G) &= \sum_{i=0}^{N-1} \sum_{j \in B_i \setminus \{i\}} \mathbb{P}[X_i = 1, X_j = 1], \end{aligned}$$

where the neighborhoods of dependence B_i are defined as in Lemma 2.2. For $S \subseteq \{1, \ldots, n\}$, denote by $i(S)$ the strategy profile in which the players of the set S play 1 and the players not in S play 0. Then writing $1(j \in B)$ for the indicator of that $j \in B$ we have:

$$\begin{aligned} b_2(G) &= \sum_{i=0}^{N-1} \sum_{j \in B_i \setminus \{i\}} \mathbb{P}[X_i = 1, X_j = 1] \\ &= \sum_{i=0}^{N-1} \sum_{j \neq i} \mathbb{P}[X_i = 1, X_j = 1] 1(j \in B_i) \\ &= N \sum_{j \neq 0} \mathbb{P}[X_0 = 1, X_j = 1] 1(j \in B_0) \qquad \text{(by symmetry)} \\ &= N \sum_{k=1}^{n} \sum_{S, |S| = k} \mathbb{P}[X_0 = 1, X_{i(S)} = 1] 1(i(S) \in B_0) \end{aligned}$$

We will bound the sum above by bounding

$$N \sum_{k=1}^{\lfloor \delta n \rfloor} \sum_{S, |S| = k} \mathbb{P}[X_0 = 1, X_{i(S)} = 1] 1(i(S) \in B_0), \tag{22}$$

and

$$N \sum_{k=\lfloor \delta n \rfloor + 1}^{n} \sum_{S, |S| = k} \mathbb{P}[X_0 = 1, X_{i(S)} = 1] 1(i(S) \in B_0) \tag{23}$$

seperately.

Note if some set S satisfies $|S| \leq \lfloor \delta n \rfloor$ then $|\mathcal{N}(S)| \geq \alpha |S|$ since the graph is a strong (α, δ)-expander. Moreover, each vertex (player) of the set $\mathcal{N}(S)$ is playing its best response to the strategies of its neighbors in both profiles 0 and $i(S)$ with probability $\frac{1}{4}$, since its environment is different in the two profiles. On the other hand, each player not in that set is in best response in both profiles 0 and $i(S)$ with probability at most

$\frac{1}{2}$. Hence, we can bound (22) by

$$N\sum_{k=1}^{\lfloor \delta n\rfloor}\sum_{S,|S|=k}\mathbb{P}[X_0=1,X_{i(S)}=1] \leq N\sum_{k=1}^{\lfloor \delta n\rfloor}\sum_{S,|S|=k}\left(\frac{1}{2}\right)^{n-\alpha k}\left(\frac{1}{4}\right)^{\alpha k}=\sum_{k=1}^{\lfloor \delta n\rfloor}\binom{n}{k}\left(\frac{1}{2}\right)^{\alpha k}$$
$$< \left(1+\left(\frac{1}{2}\right)^{\alpha}\right)^{n}-1\leq 2n^{-\epsilon}$$

To bound the second term, notice that, if some set S satisfies $|S|\geq\lfloor \delta n\rfloor+1$, then since the graph is a strong (α,δ)-expander $\mathcal{N}(S)\equiv V$ and, therefore, the environment of every player is different in the two profiles 0 and $i(S)$. Hence, $1(i(S)\in B_0)=0$. By combining the above we get that

$$b_2(G)\leq 2n^{-\epsilon}.$$

It remains to bound the easier term $b_1(G)$. We have

$$\begin{aligned}b_1(G)-2^{-n} &= \sum_{i=0}^{N-1}\sum_{j\in B_i\setminus\{i\}}\mathbb{P}[X_i=1]\mathbb{P}[X_j=1]=\sum_{i=0}^{N-1}\sum_{j\neq i}\mathbb{P}[X_i=1]\mathbb{P}[X_j=1]1(j\in B_i)\\ &= 2^{-n}\sum_{j\neq 0}1(j\in B_0)\\ &= 2^{-n}\sum_{k=1}^{\lfloor \delta n\rfloor}\sum_{S,|S|=k}1(i(s)\in B_0)+2^{-n}\sum_{k=\lfloor \delta n\rfloor+1}^{n}\sum_{S,|S|=k}1(i(s)\in B_0).\end{aligned}$$

The second term is zero as before. For all large n the first term contains at most $2^{n/2-1}$ terms and is therefore bounded by $2^{-n/2-1}$. It follows that

$$b_1(G)+b_2(G)\leq 2n^{-\epsilon}+2^{-n/2}.$$

An application of the result by Arratia et al.[5] concludes the proof. ■

3.2 A Sufficient Condition for Non-Existence of Equilibria: Indifferent Matching Pennies

In this section we provide a proof of Theorem 1.7. Recall that an edge of a graph is called *d-bounded* if both adjacent vertices have degrees smaller or equal to d. Theorem 1.7 specifies that any graph with many such edges is unlikely to have PNE. We proceed to the proof of the claim.

Proof: Consider a d-bounded edge in a game connecting two players a and b, each one Interacting with $d-1$ (or less) other players denoted by $a_1,a_2\dots a_{d-1}$ and $b_1,b_2\dots b_{d-1}$. Recall that if the game played on this edge is indifferent matching pennies then the game has no PNE. The key observation is that a d-bounded edge is an indifferent matching pennies game probability at least $p_{imp}=(\frac{1}{8})^{2^{2d-2}}$ — since in a random two player game a matching pennies appears with probability $\frac{1}{8}$ and there are 2^{2d-2} possible pure strategy profiles for the players $a_1,a_2\dots a_{d-1},b_1,b_2\dots b_{d-1}$.

Note that for a collection of edge disjoint edges, the events that they play indifferent matching pennies are independent and therefore the probability that the game has PNE is bounded by:

$$(1-p_{imp})^m\leq\exp(-mp_{imp})=\exp\left(-m\left(\frac{1}{8}\right)^{2^{2d-2}}\right).$$

For the second statement of the theorem note that if there are m bounded d edges, then there are at least $m/(2d)$ vertex disjoint bounded d edges.

The algorithmic statement follows from the fact that we may explore all edges of the graphs in complexity $\binom{n}{2}$ to find the d-bounded edges. Then in time $O(m2^{d+2})$ we can check if any of these edges play indifferent matching pennies.

The final statement has a similar proof where now the potential witnesses are the edges of $\mathcal{E}$. ■

Many random graphical games on deterministic graphs such as players arranged on a line, grid, or any other bounded degree graph (with $\omega(1)$ edges) are special cases of the above theorem and hence are unlikely to have PNE asymptotically.

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A Appendix

High Density Proof

The first function we need to bound is the following

$$S(n,p) := \sum_{s=1}^{n} \binom{n}{s} 2^{-n} \left[(1+(1-p)^s)^{n-s} - (1-(1-p)^s)^{n-s} \right].$$

To bound $S(n,p)$ we split the range of the summation into the following regions and bound the sum over each region separately. Let us choose $\alpha = \alpha(\epsilon) \in (0, 0.5)$ such that $2^{H(\alpha)} \cdot \frac{1+e^{-\frac{\epsilon}{2+\epsilon}}}{2} < c < 1$, where $H(\cdot)$ is the entropy function and c is some constant bounded away from 1, and let us define the following regions:

I. $1 \leq s < \frac{\epsilon}{(2+\epsilon)p}$;

II. $\frac{\epsilon}{(2+\epsilon)p} \leq s < \alpha n$;

III. $\alpha n \leq s < \frac{1}{2+\epsilon} n$;

IV. $\frac{1}{2+\epsilon} n \leq s < n$.

Let us then write

$$S(n,p) = S_{\mathrm{I}}(n,p) + S_{\mathrm{II}}(n,p) + S_{\mathrm{III}}(n,p) + S_{\mathrm{IV}}(n,p),$$

where $S_{\mathrm{I}}(n,p)$ denotes the sum over region I etc., and bound each term separately.

Region I.

The following lemma will be useful.

Lemma A.1 *For all* $\epsilon > 0$*,* $p \in (0,1)$ *and* s*,* $1 \leq s < \frac{\epsilon}{(2+\epsilon)p}$*,*

$$(1-p)^s \leq 1 - \frac{(2+0.5\epsilon)sp}{2+\epsilon}.$$

Proof: First note that, for all $k \geq 1$,

$$\binom{s}{2k+2} p^{2k+2} \leq \binom{s}{2k+1} p^{2k+1}. \tag{24}$$

To verify the latter note that it is equivalent to

$$s \leq 2k+1+\frac{2k+2}{p},$$

which is true since $s \leq \frac{\epsilon}{(2+\epsilon)p} = \frac{1}{(\frac{2}{\epsilon}+1)p} \leq \frac{1}{p}$.

Using (24), it follows that

$$(1-p)^s \leq 1 - \binom{s}{1}p + \binom{s}{2}p^2. \tag{25}$$

Note finally that

$$\frac{0.5\epsilon}{2+\epsilon} sp > \frac{s(s-1)}{2} p^2,$$

which applied to (25) gives

$$(1-p)^s \leq 1 - \frac{(2+0.5\epsilon)sp}{2+\epsilon}.$$

■

Using Lemma A.1, we get

$$
\begin{aligned}
S_{\mathrm{I}}(n,p) &\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-n}(1+(1-p)^s)^{n-s} \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-n}\left(1+1-\frac{(2+0.5\epsilon)sp}{2+\epsilon}\right)^{n-s} \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-s}\left(1-\frac{(1+0.25\epsilon)sp}{2+\epsilon}\right)^{n-s} \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-s} \exp\Big(-\frac{(1+0.25\epsilon)sp}{2+\epsilon}(n-s)\Big) \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-s} \exp\Big(-\frac{(1+0.25\epsilon)sp}{2+\epsilon}n\Big) \exp\Big(\frac{(1+0.25\epsilon)sp}{2+\epsilon}s\Big) \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \binom{n}{s} 2^{-s} \exp(-(1+0.25\epsilon)\log n\ s) \exp\Big(\frac{(1+0.25\epsilon)\epsilon}{(2+\epsilon)^2}s\Big) \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} n^s 2^{-s} n^{-(1+0.25\epsilon)s} \exp\Big(\frac{1}{2}s\Big) \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \left(\frac{\sqrt{e}}{2}\right)^s n^{-0.25\epsilon s} \\
&\leq \sum_{s<\frac{\epsilon}{(2+\epsilon)p}} \left(\frac{\sqrt{e}}{2}\right)^s n^{-0.25\epsilon} \\
&\leq n^{-0.25\epsilon} \sum_{s<\frac{2\epsilon}{(2+\epsilon)p}} \left(\frac{\sqrt{e}}{2}\right)^s \\
&= O(n^{-0.25\epsilon}) \qquad\qquad \left(\text{since } \frac{\sqrt{e}}{2}<1\right)
\end{aligned}
$$

Region II.

We have

$$
\begin{aligned}
S_{\mathrm{II}}(n,p) &\le \sum_{\frac{\epsilon}{(2+\epsilon)p} \le s < \alpha n} \binom{n}{s} 2^{-n}(1+(1-p)^s)^n \\
&\le \sum_{\frac{\epsilon}{(2+\epsilon)p} \le s < \alpha n} \binom{n}{s} 2^{-n}\left(1+e^{-ps}\right)^n \\
&\le \sum_{\frac{\epsilon}{(2+\epsilon)p} \le s < \alpha n} \binom{n}{\alpha n} 2^{-n}\left(1+e^{-p\frac{\epsilon}{(2+\epsilon)p}}\right)^n \\
&\le \alpha n \binom{n}{\alpha n}\left(\frac{1+e^{-\frac{\epsilon}{2+\epsilon}}}{2}\right)^n \\
&\le \alpha n 2^{nH(\alpha)}(n+1)\left(\frac{1+e^{-\frac{\epsilon}{2+\epsilon}}}{2}\right)^n \qquad \left(\text{since } \binom{n}{k} \le (n+1)2^{nH(\frac{k}{n})}\right) \\
&\le \alpha n(n+1)\left(2^{H(\alpha)} \cdot \frac{1+e^{-\frac{\epsilon}{2+\epsilon}}}{2}\right)^n = \exp(-\Omega(n)),
\end{aligned}
$$

because α was chosen so that $2^{H(\alpha)} \cdot \frac{1+e^{-\frac{\epsilon}{2+\epsilon}}}{2} < c < 1$.

Region III.

We will show that every term of the summation $S_{\mathrm{III}}(n,p)$ is exponentially small and, hence, $S_{\mathrm{III}}(n,p)$ itself is exponentially small. Indeed

$$\begin{aligned}
\binom{n}{s}2^{-n}(1+(1-p)^s)^n &\le \binom{n}{s}2^{-n}(1+e^{-ps})^n \\
&\le \binom{n}{s}2^{-n}(1+e^{-p\alpha n})^n \\
&\le \binom{n}{s}2^{-n}(1+e^{-(2+\epsilon)\alpha\log n})^n \\
&= \binom{n}{s}2^{-n}\left(1+\frac{1}{n^{(2+\epsilon)\alpha}}\right)^n \\
&= \binom{n}{s}2^{-n}\left(1+\frac{1}{n^{(2+\epsilon)\alpha}}\right)^{n^{(2+\epsilon)\alpha}n^{1-(2+\epsilon)\alpha}} \\
&\le \binom{n}{s}2^{-n}e^{n^{1-(2+\epsilon)\alpha}} \\
&\le \binom{n}{\frac{n}{2+\epsilon}}2^{-n}e^{n^{1-(2+\epsilon)\alpha}} \\
&\le (n+1)2^{nH\left(\frac{1}{2+\epsilon}\right)}2^{-n}e^{n^{1-(2+\epsilon)\alpha}} \\
&= (n+1)2^{n\left(H\left(\frac{1}{2+\epsilon}\right)-1\right)}e^{n^{1-(2+\epsilon)\alpha}} = \exp(-\Omega(n)),
\end{aligned}$$

where the last assertion follows from $H\left(\frac{1}{2+\epsilon}\right) < 1$ which is true since $\epsilon = \Omega(1)$.

Region IV.

Note that, if $xk \le 1$, then by the mean value theorem

$$(1+x)^k - (1-x)^k = \le 2x \max_{1-1/k\le y\le 1+1/k} ky^{k-1} = 2kx(1+1/k)^{k-1} \le 2ekx.$$

We can apply this for $k = n-s$ and $x = (1-p)^s$ since

$$(n-s)(1-p)^s \le (n-s)e^{-ps} \le (n-s)e^{-\frac{(2+\epsilon)\log n}{n}\frac{n}{2+\epsilon}} \le \frac{n-s}{n} \le 1.$$

Hence, $S_{\text{IV}}(n,p)$ is bounded as follows

$$\begin{aligned} S_{\text{IV}}(n,p) &\le \sum_{\frac{n}{2+\epsilon}\le s\le n}\binom{n}{s}2^{-n}2e(n-s)(1-p)^s \\ &\le 2e\cdot 2^{-n}\cdot n\sum_{\frac{n}{2+\epsilon}\le s\le n}\binom{n}{s}(1-p)^s \\ &\le 2e\cdot 2^{-n}\cdot n(1+(1-p))^n \\ &\le 2en\left(1-\frac{p}{2}\right)^n \\ &\le 2ene^{-\frac{p}{2}n} \\ &\le 2ene^{-\frac{(2+\epsilon)\log n}{2n}n} \\ &\le 2enn^{-\frac{2+\epsilon}{2}} \\ &\le 2en^{-\frac{\epsilon}{2}}. \end{aligned}$$

Putting everything together

Combining the above we get that

$$S(n,p) \le O(n^{-\epsilon/4}) + \exp(-\Omega(n))$$

.

Bounding R

The next function we need to bound is:

$$R(n,p) = \sum_{s=1}^{n}\binom{n}{s}2^{-n}\min(1, n(1-p)^{s-1})$$

We will bound it by:

$$R(n,p) \le \sum_{s=1}^{n}\binom{n}{s}2^{-n}\min(1, n\exp(-p(s-1))) \tag{26}$$

$$\le 2^{-n}\sum_{1\le s\le n/(2+\epsilon/2)+1}\binom{n}{s} + 2^{-n}\sum_{s>n/(2+\epsilon.2)+1}\binom{n}{s}n\exp(-p(s-1)) \tag{27}$$

$$\le 2^{-n}\sum_{s>n/(2+\epsilon/2)+1}\binom{n}{s}n\exp(-p(s-1)) + \exp(-\Omega(n)) \tag{28}$$

In order to bound the last sum we observe that when $s > n/(2+\epsilon/2)+1$ we have

$$n\exp(-p(s-1)) \le n\exp(-\frac{(2+\epsilon)\log n}{n}\frac{n}{2+\epsilon/2}) \le n\times n^{-(2+\epsilon)/(2+\epsilon/2)} \le n^{-\epsilon/4},$$

as needed.

Proof of Theorem ??

We first note that when c is fixed then the probability that any two edges intersect is $o(1)$. On the event that there are no intersecting edges the probability that there is PNE equals the probability that no edge of the graph defines a matching pennies game between two players. Therefore the asymptotic probability is given by:

$$(1 - p/8)^{\binom{n}{2}} = \left(1 - \frac{c}{8n^2}\right)^{\frac{n(n-1)}{2}} \longrightarrow e^{-\frac{c}{16}}.$$