Bayesian Auctions with Friends and Foes

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Abstract. We study auctions whose bidders are embedded in a social or economic network. As a result, even bidders who do not win the auction themselves might derive utility from the auction, namely, when a friend wins. On the other hand, when an enemy or competitor wins, a bidder might derive negative utility. Such spite and altruism will alter the bidding strategies. A simple and natural model for bidders' utilities in these settings posits that the utility of a losing bidder i as a result of bidder j winning is a constant (positive or negative) fraction of bidder j's utility.

We study such auctions under a Bayesian model in which all valuations are distributed independently according to a known distribution, but the actual valuations are private. We describe and analyze Nash Equilibrium bidding strategies in two broad classes: regular friendship networks with arbitrary valuation distributions, and arbitrary friendship networks with identical uniform valuation distributions.

1 Introduction

The traditional view of auctions posits that bidders only care if they win the item(s) and at what price. The utility of bidders not winning the auction is 0, regardless of the actual outcome. If the auction is conducted among perfectly rational strangers, the items are solely for resale, and no future competitive advantage is gained by winning an auction cheaply, this assumption is quite accurate. However, in many realistic scenarios, the bidders are embedded in social and economic networks, which will affect their perception of an auction's outcome.

For instance, if bidders i and j are friends, and bidder j wins the auction at a cheap price, bidder i may derive direct or indirect benefits. These could take the form of shared joy, or tangible financial benefits due to bidder j's generosity. Similarly, if bidders i and j are enemies, then bidder i might actively resent j's winning. Even in a purely economic environment, there can be differences in i's perception of the winner. For instance, if j is a direct competitor of i, then winning the auction cheaply might give bidder j significant future market advantage. Thus, bidder i derives negative utility from j's winning. If j mostly belongs to a different market, then i might be neutral to j's winning. And if i and j might be future collaborators, then i might derive some positive utility from j's winning.

These observations motivate the study of auctions in which the utility of losers is not always 0, but rather depends on the *identity of the winner, and the utility the winner derives from the auction.* We can model this setting naturally with a spite/altruism matrix $A = (a_{i,j})$, where each $a_{i,j} \in (-1, 1)$ for $i \neq j$, and $a_{i,i} \in [0, 1]$ for each i. If bidder j wins the auction and obtains subutility $\overline{u}_j$, then bidder i's utility from the auction is $a_{i,j} \cdot \overline{u}_j$. Thus, if $a_{i,j} > 0$, then bidder i is *altruistic* toward bidder j (or a *friend*); if $a_{i,j} < 0$, then bidder i is *spiteful* toward bidder j (or a *foe*)[1]. Notice that we do not assume A to be symmetric.

Auctions with spite among bidders have been studied before [6, 7, 16, 18, 21]. However, in all past work, the assumption was that each off-diagonal entry of the matrix A was the same (and negative), i.e., all bidders have the same spite level toward each other. We call this the case of *uniform spite*. While it is interesting as an analysis of the effects of general distrust or future competition between bidders, it does not take into account the effects of social or economic networks on individual behaviors.

In this paper, we study auctions with Bayesian priors in the presence of more general altruism/spite matrices. For two large subclasses of these auctions, we explicitly describe a Nash Equilibrium. These two subclasses are the following:

1. The spite/altruism matrix A is arbitrary, but each bidder's valuation of the item is drawn independently and uniformly from the interval $[0, 1]$, and the auction is first-price.
2. The valuations are drawn independently from $[0, 1]$ according to an arbitrary (but identical) distribution for all bidders, and the social network of bidders is *regular*. This means that each bidder i has non-zero $a_{i,j}$ for the same number d of other bidders j, and all such non-zero entries have the same value $a_{i,j} = a$. In this case, we analyze both first- and second-price auctions.

[1] Past work [6, 7] used $a_{i,j} > 0$ for spite. However, the convention we adopt here simplifies notation, and is consistent with the notation of [14, 9, 17].

These characterizations significantly generalize recent results of Morgan et al. [18] and Brandt et al. [6], which characterized a Nash Equilibrium for the case of uniform spite. We also point out here that the equilibrium in the first case is not symmetric: different bidders have different bidding strategies. This is significant in particular from a technical viewpoint, as past analysis has relied heavily on symmetry assumptions in order to be able to derive equilibrium strategies.

Our explicit characterization allows us to derive several interesting corollaries. For the case of arbitrary social networks and uniform valuation distributions, the explicit characterization allows us to study how changes in the social or economic network affect bidding behavior. Perhaps somewhat surprisingly, an increase in spite does not always lead to an increase in bids. Instead, we show that whether it leads to an increase or decrease in bids depends on whether the recipient of spite is currently overbidding or underbidding.

A further corollary concerns auctions with several cliques of friends who are indifferent to other cliques. Our characterization allows us to easily derive explicit equilibrium bidding strategies in this case. Interestingly, the strategy of the members of a clique depends only on the size of the clique, the strength of ties in the clique, and the total number of bidders, but not on the strength of the ties in other cliques. In this case in particular, the resulting bidding strategies can be considered an alternative to collusion without explicit information exchange between the members of a clique.

For the case of regular social networks and uniformly random valuations, we show that if $a < 0$ (i.e., bidders only have foes and neutral other bidders), the expected revenue of the second-price auction dominates the first-price auction. Conversely, if $a > 0$, then the expected revenue of the first-price auction dominates the second-price auction. (The case of $a = 0$ corresponds exactly to standard auctions, and the Revenue Equivalence Theorem implies that both auctions provide the same revenue.)

1.1 Related Work

The notion of *spite* and *altruism* as defined here broadly falls into the class of *allocation externalities* in auctions: the utility of a bidder depends not exclusively on her own allocation, but also on the allocations of other bidders. There is a large amount of literature on various types of allocation externalities (see, e.g., [11–13, 8]). In particular, Jehiel et al. [12] construct revenue-maximizing auctions for the case where each potential buyer has a given constant externality depending on the identity of the winner. Thus, the difference to our model is that in the model of [12], a loser's utility does not depend on the *price* at which the winner obtained the object, only the winner's identity.

Altruism and spite specifically in the context of auctions were studied in several recent papers: Brandt and Weiß[7] studied full-information equilibria between two bidders, both of whom have spite level $a = \frac{1}{2}$. Morgan et al. [18] and Brandt et al. [6] focused on Bayesian Nash Equilibria of first-price and second-price auctions with uniform spite. The results in these two papers are very similar to each other, and differ mostly in the precise model of the utility of the winner, as discussed briefly in Section 2. Vetsikas and Jennings [21] extend this work to auctions for multiple items, still assuming uniform spite among the bidders.

A similar model is also studied in a recent paper by Deng and Qi [10] on auction design for pricing priority rights. Losers in this model also incur a negative utility, albeit one that depends on their own utility for the item, rather than the winner's. The goal in [10] is to design a truthful, egalitarian and budget-balanced auction.

Several recent papers have analyzed the impact of spiteful or altruistic behavior in other game-theoretic settings. In the context of congestion games, [9] shows that when all players are at least β-altruistic (meaning that their utility is a convex combination of their own latency and the derivative of the average latency, with weight β on the average latency), then the PoA is bounded by $1/\beta$. Babaioff et al. [3] and Roth [20] consider the effect of malicious or Byzantine players on the PoA or regret.

In the context of network inoculation [2], Moscibroda et al. [19] study the effect of Byzantine malicious players, while Meier et al. [17] show that friendship with neighbors can sometimes lead to significantly more efficient network inoculations.

The impact of social network structure on games has also recently been studied by Ashlagi et al. [1], under the name *social context games*. They posit that the utility of an agent can be computed from the subutility functions in her neighborhood, according to various competitive or collaborative aggregation functions. The specific games studied in [1] differ from the auctions considered here, and mostly belong to the class of resource selection games.

2 Model and Preliminaries

Each of the n bidders has a valuation v_i drawn independently from *the same* distribution F over $[0, 1]$. While the valuations are private, F is common knowledge among all bidders. We identify the distribution with its cumulative distribution function (cdf), and use $f = F'$ to denote its density function.

We study auctions in which the auctioneer is selling a single item to spiteful and altruistic bidders. Bids are denoted by b_i. The auction mechanism selects as winner a bidder w maximizing b_w (breaking ties arbitrarily, but consistently). We define the *threshold bid* of w to be $\tau_w = \max_{j \neq w} b_j$. In a first-price auction, bidder w pays b_w, while in a second-price auction, she pays τ_w.

In a second-price auction, the *subutility* of the winning bidder is $\overline{u}_w = v_w - \tau_w$. Similarly, for a first-price auction, the subutility of the winning bidder is $\overline{u}_w = v_w - b_w$. In both cases, the subutility of all losing bidders is $\overline{u}_i = 0$.

In an auction with altruism and spite, the *utility* of a bidder is a combination of her own and the other bidders' subutilities. (A similar model was proposed by Ledyard [14].) Specifically, for any bidder i:

$$u_i = \textstyle\sum_j a_{i,j} \cdot \overline{u}_j. \tag{1}$$

Since it is reasonable to assume that each bidder cares more about her own subutility than that of others, we assume that $|a_{i,j}| < a_{i,i}$ for all i, j. Substituting the specific subutilities of first-price and second-price auctions into Equation (1), we obtain the following utilities for bidders i.

$$\text{First-price auction:} \qquad u_i = \begin{cases} a_{i,i} \cdot (v_i - b_i) & \text{for } i = w \\ a_{i,w} \cdot (v_w - b_w) & \text{for } i \neq w \end{cases} \tag{2}$$

$$\text{Second-price auction:} \qquad u_i = \begin{cases} a_{i,i} \cdot (v_i - \tau_i) & \text{for } i = w \\ a_{i,w} \cdot (v_w - \tau_w) & \text{for } i \neq w \end{cases} \tag{3}$$

Remark 1. The definition used by Brandt et al. [6] is the special case of our definition when $a_{i,j} = a < 0$ for all $i \neq j$, i.e., players have uniform spite, and $a_{i,i} = 1 + a$. (However, [6] uses $a > 0$ for spite.) The definition of Morgan et al. [18] is nearly identical, except it corresponds to the case of $a_{i,i} = 1$ for all i.

Bidders are assumed to maximize expected utility, and may submit bids $b_i \neq v_i$. Specifically, we denote the bid function for bidder i by $b_i(\cdot)$, meaning that with valuation v, bidder i will submit a bid of $b_i(v)$. We stress here that while the valuations are private, both the common distribution of valuations and the altruism/spite matrix A are common knowledge. (We briefly discuss the latter point in Section 4.) The externalities in our setting arise solely from the perception that losers have of the winner; there is no correlation between the valuations of different bidders.

According to Equation (1), the larger $|a_{i,j}|$ (and the smaller $a_{i,i}$), the more important the winning or losing of other bidders becomes to i. Notice, however, that we do not recursively consider the utility a bidder derives from another bidder's perceived utility. Such *systems of utility functions* are studied, for instance, by Bergstrom [4], who shows that by solving a system of linear equations, they can be reduced to the case studied here.

While our model allows for friendship between bidders, we assume that the bidders do not collude. Collusion would require the bidders to share their private valuations with each other, which may be an inferior strategy in terms of the individual utilities. Furthermore, it is not clear how the profit should be split between colluding bidders when $0 < a_{i,j} < 1$.

3 Calculating and Analyzing Equilibria

We first derive general Nash Equilibrium conditions for arbitrary distributions F on $[0, 1]$ and altruism, for both first- and second-price auctions. Since these conditions are too complicated to solve in general, we then focus on two special cases:

1. First-price auctions with arbitrary spite/altruism matrices A, but in which all valuations are drawn *uniformly* from the interval $[0, 1]$. For this case, we present a (non-symmetric) Nash Equilibrium, and show how the bidding strategies change if the entries of A change.
2. Networks in which each bidder has the same number d of acquaintances, and feels the same spite/altruism level a toward all of them. Thus, we have a social network in which each node has outdegree d, and all bidders have uniform spite/altruism. For this case, we analyze both first- and second-price auctions under arbitrary distributions F of valuations. We show that under uniformly random valuations, the revenue of the second-price auction dominates the first-price auction for $a < 0$, while the domination is reversed for $a > 0$.

We denote by b_i^{-1} the inverse function of the bidding function, i.e., $b_i^{-1}(b)$ is the valuation v such that bidder i with valuation v would bid b.[2]

[2] We are thus implicitly assuming that the bidding functions are strictly increasing.

Lemma 1. *Assume that all valuations are drawn independently from the same distribution F over $[0,1]$.*

1. *Nash Equilibria of first-price auctions satisfy the following system of differential equations:*

$$\sum_{j\neq i} \Big(a_{i,i}(v-b_i(v)) + a_{i,j}(b_i(v) - b_j^{-1}(b_i(v)))\Big) \cdot \frac{f(b_j^{-1}(b_i(v)))\cdot b_j^{-1\prime}(b_i(v))}{F(b_j^{-1}(b_i(v)))} = a_{i,i}. \tag{4}$$

2. *Nash Equilibria of second-price auctions satisfy the following system of differential equations:*

$$\begin{aligned}
&a_{i,i} \cdot \sum_{j\neq i} \frac{f(b_j^{-1}(b_i(v))\cdot b_j^{-1\prime}(b_i(v)))}{F(b_j^{-1}(b_i(v)))} \cdot (v - b_i(v)) \\
&+ \sum_{j\neq i} \frac{a_{i,j}}{F(b_j^{-1}(b_i(v)))} \cdot \Big(- F(b_j^{-1}(b_i(v))) \cdot b_i(v) \cdot \sum_{k\neq i,j} \frac{f(b_k^{-1}(b_i(v)))b_k^{-1\prime}(b_i(v))}{F(b_k^{-1}(b_i(v)))} \\
&\qquad + b_i(v) \cdot \sum_{\ell\neq i} f(b_\ell^{-1}(b_i(v))) \cdot b_\ell^{-1\prime}(b_i(v)) \\
&\qquad - b_j^{-1}(b_i(v)) \cdot f(b_j^{-1}(b_i(v))) \cdot b_j^{-1\prime}(b_i(v)) - 1\Big) \\
&= - \sum_{j\neq i} a_{i,j}
\end{aligned} \tag{5}$$

The proof is rather technical, and deferred to the full version of this paper due to space constraints. Compared to standard analysis of equilibrium strategies in auctions, it requires deriving a system of differential equations, rather than a single differential equation. In general, this system of differential equations (4) or (5) does not admit a direct solution, due to the interplay between inverses of bidding functions. We therefore next focus on special cases where the particular form of bidding functions allows us to simplify the differential equations further.

3.1 First-Price Auctions with Uniform Valuations

Our first special case is that of first-price auctions with uniform valuations on $[0,1]$, i.e., $F(x) = x$ for $x \in [0,1]$. In this case, we can calculate a Bayesian Nash Equilibrium explicitly, because there happens to be a Nash Equilibrium where each bidder bids $b_i(v) = \gamma_i v$ for some constant γ_i. Unfortunately, a guess of $b_i(v) = \gamma_i v$, or even $b_i(v) = \gamma_i v + \xi_i$, does not appear to lead to a solution of the corresponding system (5) for second-price auctions, and we are not aware of any explicit characterization of an equilibrium of the second-price auction here.

Theorem 1. *Assume that all valuations are drawn independently and uniformly from $[0,1]$. There is a Bayesian Nash Equilibrium for first-price auctions with an arbitrary friendship/spite matrix A where each bidder i bids $b_i(v_i) = \gamma_i v_i$, with*

$$\gamma_i = \frac{\det(C)}{\det(C) - \det(C_i)}.$$

The matrix C has entries $c_{i,i} = -(n-1)$ and $c_{i,j} = \frac{a_{i,j}}{a_{i,i}}$ for $i \neq j$, and C_i is formed by replacing the i^{th} column of C by all 1's.

Proof. We start with the general system of differential equations derived as Equation (4). We now substitute that $F(x) = x$ and $f(x) = 1$ for all $x \in [0, 1]$, obtaining that

$$\sum_{j \neq i} \Big(a_{i,i}(v - b_i(v)) + a_{i,j}(b_i(v) - b_j^{-1}(b_i(v))) \Big) \cdot \frac{b_j^{-1\prime}(b_i(v))}{b_j^{-1}(b_i(v))} = a_{i,i}.$$

We next guess that $b_i(v) = \gamma_i v$ for each bidder i, i.e., each bidder simply scales her valuation by a constant factor that may depend on A, but not on the valuations. Then, $(b_j^{-1})'(b_i(v)) = 1/\gamma_j$, and $b_j^{-1}(b_i(v)) = \frac{\gamma_i}{\gamma_j} \cdot v$, so we obtain

$$\sum_{j \neq i} \Big(a_{i,i}(1 - \gamma_i)v + a_{i,j}(\gamma_i v - \frac{\gamma_i}{\gamma_j} \cdot v) \Big) \cdot \frac{1}{\gamma_i v} = a_{i,i}.$$

Canceling all v terms and the γ_i, and pulling constant terms out of the sum, this simplifies to

$$(n-1)a_{i,i}(\tfrac{1}{\gamma_i} - 1) + \sum_{j \neq i} a_{i,j}(1 - \tfrac{1}{\gamma_j}) = a_{i,i}.$$

Writing $\beta_i = 1 - \frac{1}{\gamma_i}$, this system becomes $-(n-1)\beta_i + \sum_{j \neq i} \frac{a_{i,j}}{a_{i,i}} \beta_j = 1$ for all i. Thus, the vector $\boldsymbol{\beta}$ of all β_i entries solves $C \cdot \boldsymbol{\beta} = \mathbf{1}$, where $\mathbf{1}$ is the n-dimensional all-ones vector. The theorem now follows from Cramer's rule, which gives that $\beta_i = \frac{\det(C_i)}{\det(C)}$. Because $|a_{i,j}| < a_{i,i}$ for all i, j, all off-diagonal entries of C are strictly less than 1, so C is diagonally dominant. Gershgorin's Disc Theorem thus guarantees that $\det(C) \neq 0$, and the system always has a solution. ■

Competing Cliques. One natural special case which can be solved easily using our general result in Theorem 1 is that of disjoint cliques of friends in an auction. The bidders form g disjoint groups $S_1, \ldots, S_g$. Within group S_k, all bidders have altruism $a^{(k)}$ to each other (and $a_{i,i} = 1$). Across groups, bidders are indifferent, i.e., a bidder's altruism or spite level towards any other bidder who is not in his group is 0. Then, C is a block matrix, and the system of linear equations can be solved for each block separately. Due to symmetry, within each group S_k, all bidders will use the same bidding strategy, i.e., $\beta_i = \beta_j =: \beta^{(k)}$ whenever $i, j \in S_k$. The linear equality thus simplifies to $-(n-1)\beta^{(k)} + (|S_k| - 1) \cdot a^{(k)} \cdot \beta^{(k)} = 1$, with the solution $\beta^{(k)} = \frac{1}{(|S_k|-1) \cdot a^{(k)} - (n-1)}$. Substituting this into the definition of γ_i, we obtain the following corollary:

Corollary 1. *If the bidders form disjoint cliques S_k with mutual altruism $a^{(k)}$, and all valuations are drawn uniformly from $[0, 1]$, then there exists a Bayesian Nash Equilibrium in which each bidder $i \in S_k$ bids $\frac{n-1-a^{(k)}(|S_k|-1)}{n-a^{(k)}(|S_k|-1)} \cdot v_i$.*

Notice that this corollary reveals several interesting tendencies. First, both β_i and γ_i are always less than 1, and decreasing in $|S_k|$ and $a^{(k)}$. This is not entirely unexpected, as bidders in large or tightly knit cliques feel less of a need to win the auction themselves, since they are more likely to derive utility from a friend's winning. What is perhaps more surprising is that the bidding strategy of a clique S_k does not depend on how large or tightly knit another group $S_{k'}$ is. While this follows readily from our general result, it is not at all apparent a priori, since another tightly knit group might bid lower, allowing group S_k to lower its bids safely as well.

Altruism Changes. We can also use Theorem 1 for an investigation of how bidder i's strategy changes if her spite level $a_{i,j}$ toward another member of the network changes. One would intuitively expect that if bidder i's altruism toward bidder j increases, then bidder i will always bid lower, i.e., decrease γ_i, because she derives more utility from bidder j's winning. (Indeed, for disjoint cliques, this intuition is borne out.) It turns out that this is not always the case. In response to the change of one $a_{i,j}$, the entire network's strategies adapt, and in some cases, this means that bidder i will increase her bid. The following theorem characterizes the change — its proof is quite tedious and technical, and deferred to the full version of this paper due to space constraints.

Theorem 2. *The derivative of β_i with respect to $a_{i,j}$ is*

$$\frac{\partial \beta_i}{\partial a_{i,j}} = -\frac{\det(C_{i,i}) \cdot \det(C_j)}{\det(C)^2} = -\frac{\det(C_{i,i})}{\det(C)} \cdot \beta_j,$$

where $C_{i,i}$ is the $(n-1) \times (n-1)$ matrix obtained by removing the i^{th} row and column from C, and C_j is the matrix formed by replacing the j^{th} column of C by all ones.

In the result of Theorem 2, $\det(C_{i,i})$ and $\det(C)$ always have opposite signs, because all their eigenvalues are negative, and one of them has even, the other one odd, dimension. Thus, if bidder i increases her altruism level to another bidder j who is overbidding ($\beta_j > 0$, thus $\gamma_j > 1$), then bidder i will increase her bid. Conversely, if she increases her altruism level to another bidder j who is currently underbidding, then bidder i will decrease her bid. Thus, the current bidding strategy of bidder j captures enough information to determine the direction of the change in bidder i's bid when her altruism or spite changes.

3.2 Regular Networks

In order to be able to solve the system of differential equations, we assumed in the previous section that the valuations were drawn uniformly from $[0, 1]$. As a first step towards avoiding this assumption, we consider *regular networks*, in which each node has the same out-degree d. Furthermore, we assume that for each pair of bidders (i, j) with a directed edge from i to j, the spite level is the same, $a_{i,j} = a$ for all i, j with an edge. Similarly, all diagonal entries are the same, i.e., $a_{i,i} = \alpha$ for all i.

Under this scenario, both the first-price and second-price auction have a symmetric Bayesian Nash Equilibrium, i.e., a Nash Equilibrium in which all bidding functions are the same, $b_i = b$ for all i.

Theorem 3. *For $\alpha \neq 0$, there exists a Bayesian Nash Equilibrium for first-price auctions in which all bidders bid $b(v) = \mathrm{E}[X \mid X < v]$, where X is a random variable with cdf $F(x)^{n-1-da/\alpha}$.*

Proof. Substituting the symmetric guess into the the system (4) for first-price auctions, we can simplify to

$$\sum_{j \neq i} \Big(a_{i,i}(v - b(v)) + a_{i,j} b(v) - a_{i,j} v \Big) \cdot \frac{f(v)}{F(v) b'(v)} = a_{i,i},$$

and, using the network structure, simplify further to

$$\Big(((n-1)\alpha - da)\cdot(v-b(v))\Big)\cdot\tfrac{f(v)}{F(v)b'(v)} = \alpha.$$

Solving for $b(v)$ gives us $b(v) = v - \frac{1}{n-1-da/\alpha}\cdot\frac{F(v)b'(v)}{f(v)}$. This differential equation has solution

$$b(v) = F(v)^{-(n-1-da/\alpha)}\cdot\textstyle\int_0^v x\cdot(n-1-da/\alpha)\cdot F(x)^{n-2-da/\alpha}f(x)dx. \qquad (6)$$

Thus, we have proved the theorem. ■

Note that the bidding function can be interpreted as the expectation of the highest of $(n-1)-\frac{ad}{\alpha}$ private values below v, in spite of the fact that $(n-1)-\frac{ad}{\alpha}$ may be a fractional number. Notice that this theorem does not characterize *all* equilibria, and indeed, it seems very likely that this auction also possesses asymmetric Nash Equilibria (see also the discussion in [6]).

Substituting the uniform distribution over $[0,1]$ for every bidder's valuation, we obtain the following corollary:

Corollary 2. *There is a Bayesian Nash Equilibrium for first-price auctions with all valuations uniformly distributed in* $[0,1]$ *in which all bidders bid* $b(v) = (1-\frac{\alpha}{n\cdot\alpha - ad})\cdot v$.

In particular, when $d = n-1$, Theorem 3 and Corollary 2 recover the results of Brandt et al. [6] who showed that $b(v) = \frac{n-1}{n+a}\cdot v$ for uniform spite levels (with $\alpha = 1+a$), and those of Morgan et al. [18] (with $\alpha = 1$).

Second-Price Auctions. We next turn our attention to second-price auctions, and prove the following theorem.

Theorem 4. *For* $a \neq 0$, *there is a Bayesian Nash Equilibrium for the second-price auction with regular friendship graphs in which all bidders bid* $b(v) = \mathrm{E}[X \mid X > v]$, *where* X *is a random variable with cdf* $1-(1-F(x))^{1-\frac{(n-1)\alpha}{ad}}$.

Proof. We again substitute the symmetric guess $b_i = b$ for all i into the system (5), canceling and simplifying it to

$$a_{i,i}\cdot\textstyle\sum_{j\neq i}\frac{f(v)}{F(v)b'(v)}\cdot(v-b(v))$$
$$+\textstyle\sum_{j\neq i}\frac{a_{i,j}}{F(v)}\cdot\Big(-b(v)\sum_{k\neq i,j}\frac{f(v)}{b'(v)} + b(v)\sum_{\ell\neq i}\frac{f(v)}{b'(v)} - v\frac{f(v)}{b'(v)} - 1\Big) = -\sum_{j\neq i}a_{i,j}.$$

Noting that the two sums inside the parentheses almost cancel out, pulling constant terms out of the sum, and using that $\sum_{j\neq i}a_{i,j} = da$ and $a_{i,i} = \alpha$ for all i, we simplify further to

$$\alpha\cdot(n-1)\cdot\tfrac{f(v)}{F(v)b'(v)}\cdot(v-b(v)) - \tfrac{f(v)}{F(v)b'(v)}\cdot da\cdot(v-b(v)) = -(1-\tfrac{1}{F(v)})\cdot da.$$

Rearranging yields the differential equation $b(v) = v + \frac{-ad\cdot(1-F(v))\cdot b'(v)}{((n-1)\alpha - ad)\cdot f(v)}$, which for $a \neq 0$ has the solution

$$b(v) = \frac{1}{(1-F(v))^{1-\frac{(n-1)\alpha}{ad}}}\cdot\int_v^1 x\cdot(1-\tfrac{(n-1)\alpha}{ad})\cdot(1-F(x))^{-\frac{(n-1)\alpha}{ad}}f(x)dx.$$

Thus, we have proved the theorem. ■

(Note that for $a = 0$, the differential equation simplifies to $b(v) = v$, which matches the known truthful bidding strategy for standard second-price auctions.) The bidding function can be interpreted as the expectation of the lowest of $1 - \frac{(n-1)\alpha}{ad}$ private values above v. Substituting the uniform distribution over $[0, 1]$ for F gives us the following corollary:

Corollary 3. *There is a symmetric Bayesian Nash Equilibrium for the second-price auction with all bids independently and uniformly drawn from* $[0, 1]$ *in which all bidders bid*

$$b(v) = (1 + \tfrac{ad}{(n-1)\alpha - 2ad}) \cdot v - \tfrac{ad}{(n-1)\alpha - 2ad}.$$

Again, when $d = n-1$, Theorem 4 and Corollary 3 subsume the results for second-price auctions with uniform spite by Brandt et al. [6] who showed that $b(v) = \frac{v-a}{1-a}$ (with $\alpha = 1 + a$), and those by Morgan et al. [18] (with $\alpha = 1$). By combining Corollaries 2 and 3, we can compare the expected revenues of the first-price auction and second-price auction when all valuations are drawn from the uniform distribution.

Theorem 5. *Assume that the social graph is regular, with uniform spite/friendship values* $a < \alpha$, *and that the valuations of all bidders are drawn independently and uniformly from* $[0, 1]$. *Then,*

1. *In the presence of uniform spite* ($a < 0$)*, the expected revenue of the second-price auction dominates the expected revenue of the first-price auction.*
2. *In the presence of uniform altruism* ($a > 0$)*, the expected revenue of the first-price auction dominates the expected revenue of the second-price auction.*

Proof. Let b_F and b_S denote the bidding functions for first- and second-price auctions, respectively. Also, let $V_{(1)}$ and $V_{(2)}$ be the highest and second-highest valuations among all bidders, respectively. Notice that because all bidders use the same bidding function, the highest valuation always corresponds to the highest bid, and the second-highest valuation to the second-highest bid.

The revenue of the first-price auction is thus $b_F(V_{(1)})$, while the revenue of the second-price auction is $b_S(V_{(2)})$. Notice that both bidding functions are linear, so we can use linearity of expectations. Furthermore, $\mathrm{E}\left[V_{(1)}\right] = \frac{n}{n+1}$, and $\mathrm{E}\left[V_{(2)}\right] = \frac{n-1}{n+1}$. Substituting these in the bidding functions of Corollaries 2 and 3,

$$\begin{aligned}
\mathrm{E}\left[b_F(V_{(1)})\right] &= (1 - \tfrac{\alpha}{n\cdot\alpha - ad}) \cdot \tfrac{n}{n+1} = \tfrac{(n-1)\alpha - ad}{n\cdot\alpha - ad} \cdot \tfrac{n}{n+1}, \\
\mathrm{E}\left[b_S(V_{(2)})\right] &= (1 + \tfrac{ad}{(n-1)\alpha - 2ad}) \cdot \tfrac{n-1}{n+1} - \tfrac{ad}{(n-1)\alpha - 2ad} \\
&= \tfrac{(n-1)\alpha - ad}{(n-1)\alpha - 2ad} \cdot \tfrac{n-1}{n+1} - \tfrac{ad}{(n-1)\alpha - 2ad}.
\end{aligned}$$

The difference is

$$\begin{aligned}
&\mathrm{E}\left[b_S(V_{(2)})\right] - \mathrm{E}\left[b_F(V_{(1)})\right] = b_S(\mathrm{E}\left[V_{(2)}\right]) - b_F(\mathrm{E}\left[V_{(1)}\right]) \\
&= \tfrac{(n-1)\alpha - ad}{(n-1)\alpha - 2ad} \cdot \tfrac{n-1}{n+1} - \tfrac{ad}{(n-1)\alpha - 2ad} - \tfrac{(n-1)\alpha - ad}{n\cdot\alpha - ad} \cdot \tfrac{n}{n+1} \\
&= \tfrac{(n-1)\cdot(n\cdot\alpha - ad)((n-1)\alpha - ad) - n\cdot((n-1)\alpha - 2ad)\cdot((n-1)\alpha - ad)}{(n+1)\cdot((n-1)\alpha - 2ad)\cdot(n\cdot\alpha - ad)} - \tfrac{ad}{(n-1)\alpha - 2ad} \\
&= \tfrac{-ad\cdot(n\cdot\alpha - ad)}{((n-1)\alpha - 2ad)\cdot(n\cdot\alpha - ad)} - \tfrac{-\alpha(n-1)ad + (ad)^2}{((n-1)\alpha - 2ad)\cdot(n\cdot\alpha - ad)} \\
&= \tfrac{-ad\cdot\alpha}{((n-1)\cdot\alpha - 2ad)\cdot(n\cdot\alpha - ad)}.
\end{aligned}$$

Because $\alpha > 0$ by definition, the denominator is positive for all values of d and all $a \in (-1, 1)$. The numerator has the opposite sign of a. Thus, we have proved the claim. ■

Notice that Theorem 5 recovers a special case of Theorem 3 from [6] for $\alpha = 1 + a$ and $a < 0$. However, [6] proved the result for arbitrary valuation distributions, while ours holds only for uniform valuations. The techniques used in [6] do not carry over immediately when the degree d of agents is small, and generalizing Theorem 5 to arbitrary distributions is ongoing work.

4 Conclusions

In this paper, we studied auctions with spite and altruism among bidders. We gave explicit characterizations of Nash Equilibria for first-price auctions with valuations drawn uniformly from $[0, 1]$ and arbitrary spite/altruism matrices A, and for first- and second-price auctions with arbitrary valuations and regular social networks.

Many questions remain for future work. For Bayesian auctions, can we find a Nash Equilibrium for second-price auctions in general? It appears that this is significantly more complex: the fact that first-price auctions had a Nash Equilibrium in which each bidder simply multiplies her bid by a constant was fortuitous. Also, can we extend the analysis of first-price auctions to other distributions, or to priors that are not identical for different bidders? Even within the realm we considered, it would be interesting to characterize *all* Nash Equilibria, although this has proven to be quite difficult even in simpler settings.

Having characterized the Nash Equilibrium bidding strategies, we would also like to explicitly compute the revenue and social welfare of the auction. The main obstacle here is to find the expected value of the winning bid, which is now a maximum among n values drawn from different distributions. A secondary problem is that the range of each distribution ($[0, \gamma_i]$) is only given by a formula involving determinants. Calculating the revenue or social welfare would let us characterize a "price of spite" or "benefit of altruism".

Another intriguing question is whether agents can learn equilibrium bidding strategies using a natural algorithm. Assuming that each agent knows the entire matrix A is certainly unrealistic. Are there simple strategies (in the style of [5]) wherein each bidder adapts her bidding strategy based on the utility derived from earlier auctions?

Finally, we would like to extend these results beyond single-item auctions to more complex settings. A particularly promising direction would be the context of keyword auctions [15], as well as various combinatorial settings.

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