Switching Problem and Related System of Reflected Backward SDEs*

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Abstract

This paper studies the switching problem under Knightian uncertainty and recursive utilities. We show that the lower price of a plant submitted to the decisions of switching is given by the solution of a system of reflected backward stochastic differential equations (RBSDEs for short). The main feature of this system is that its components are interconnected through both the generators and the obstacles. Such a system can also be interpreted as a game problem where all the players are "partners". We prove existence, uniqueness, and stability of the solution of the RBSDE, and give the expression of the optimal strategy for the original switching problem via a verification theorem.

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1 Introduction

We first introduce through an example the standard starting and stopping (or *switching*) problem which has attracted a lot of interests during the last decades.

Assume that a power plant produces electricity whose selling price fluctuates and depends on many factors such as consumer demand, oil prices, weather and so on. It is well known that electricity cannot be stored (or too expensive to store) and once produced it should be consumed almost immediately. Therefore electricity is produced only when there is enough profitability in the market. Otherwise the power station is closed till the time when the profitability is coming back again. Then for this power station there are two modes: operating and closed. Accordingly, a management strategy of the station is an increasing sequence of stopping times $\delta = (\tau_n)_{n\geq 0}$ with $\tau_0 \triangleq 0$. At time τ_n, the manager switches the mode of the station from its current one to the other. Such a switch of modes is not free and generates expenditures.

Suppose now that we have an adapted stochastic process $X = (X_t)_{0\leq t\leq T}$ which stands for either the market electricity price or factors which determine the price. When the power station is run under a strategy $\delta = (\tau_n)_{n\geq 0}$, its yield is given by a quantity denoted by $J(\delta)$, which depends also on X and many other parameters such as utility functions, expenditures, etc. Therefore the main problem is to find an optimal management strategy $\delta^* = (\tau_n^*)_{n\geq 1}$ such that for any other δ we have $J(\delta^*) \geq J(\delta)$, *i.e.* $J(\delta^*) = \sup_\delta J(\delta)$. Once determined, the strategy δ^* gives the optimal way of running the power plant and, as a by-product, the value $J(\delta^*)$ is nothing but the fair price of the power plant in the energy market.

The two-mode starting and stopping problems have received strong attention in the literature , see, e.g., [2, 3, 4, 8, 9, 11, 12, 13, 14, 19, 21, 22, 28, 31, 35, 34, 36], ... and the references therein. In particular, Hamadène and Jeanblanc [21] study a finite horizon problem when the price X is an arbitrary process adapted to a Brownian filtration. Porchet et *al*. [31] consider exponential utilities and allow the manager to invest in a financial market. Djehiche and Hamadène [9] study a model which integrates the risk of abandonment of the economic project. Hamadène and Hdhiri [22] extends the model to the case where the price process X is adapted to a filtration generated by a Brownian motion and an independent Poisson process.

We note that this switching problem has also been used to model industries like copper or aluminium mines,..., where parts of the production process are temporarily reduced or shut down when e.g. fuel, electricity or coal prices are too high to be profitable from running them. A further area of applications includes Tolling Agreements (see Carmona and Ludkovski [6] and Deng and Xia [8] for more details).

A natural extension of the two-mode problem is the multi-mode switching problem. This has been recently studied by several authors amongst we quote Carmona and Ludkovski [6], Djehiche et al. [10] and Porchet et *al*. [32].

All the works quoted above assume that future uncertainty of market conditions is characterized by a certain probability measure P over the states of nature. That is, the distribution of the price process X is given. The Knightian uncertainty introduced by F.H. Knight [27] assumes instead that the market evolves according to one of many possible probabilities P^u, $u \in \mathcal{U}$, but we do not know which one it is. The notion of ambiguity follows similar idea, see, e.g. Chen-Epstein [7].

Our first goal of the paper is to price the power plant in such a market with Knightian uncertainty. Let $J(\delta, u)$ denote the value of the plant if the actual market probability is P^u and the manager follows strategy δ. We consider the worst scenario:

$$J^* \stackrel{\triangle}{=} \sup_\delta J(\delta) \stackrel{\triangle}{=} \sup_\delta \inf_u J(\delta, u). \tag{1.1}$$

That is, in the worst case the plant is worth at least J^*. We aim to find the value J^* and the optimal δ^* such that $J^* = J(\delta^*)$.

We note that, due to the uncertainty on u, the problem is similar to the incompleteness in financial markets, and J^* corresponds to the sub-hedging price, or say, the minimum seller's price.

We solve the problem by using Reflected Backward SDEs (RBSDEs for short). Under certain conditions on X and u and for any given strategy δ, we first show that the value $J(\delta)$ is the solution of a BSDE. The optimization over δ is an optimal stopping problem and naturally leads to RBSDEs. However, this is a two dimensional RBSDE with oblique reflections. There are very few results on this kind of RBSDEs in the literature. Via a verification theorem, we show that the solution of the RBSDE gives the quantity J^* and the optimal strategy δ^*.

We next extend our results to a very general multi-dimensional RBSDE of which both the generators and the obstacles are interconnected. While multi-dimension is obviously motivated by multi-mode switching problems, the feature that the plant's value (or price) enters the generator (or the utility function) can be interpreted as recursive utilities, see, e.g. Duffie-Epstein [15] and [16]. Our model can also be interpreted as a game problem, where the players' utilities affect each other and consequently the generators are interconnected. Moreover, under certain monotonicity conditions on the generators, which we will assume, we see that the players are "partners" with their interests positively correlated, and thus they can choose strategy δ as a group and maximize their utilities together. The general non-zero sum game problem is much more difficult. We have some result on the existence of equilibrium in a different framework, see Hamadène and Zhang [23].

We prove the existence of solutions of our RBSDEs by using the notion of the smallest g-supermartingales introduced by Peng [29] and Peng and Xu [30]. This notion can be understood as an a nonlinear version of the snell envelope. We prove the uniqueness by a verification theorem. However, for our general case the optimal strategy does not exist, so we can only obtain approximately optimal strategies. This requires some sophisticated estimates and is in fact the main technical part of the paper. As an intermediary result we obtain some stability result for high dimensional RBSDEs, which is interesting in its own rights.

The idea of using RBSDEs in starting and stopping problems appeared already in Hamadène and Jeanblanc [21]. In their model the two dimensional RBSDE is linear and can be transformed into a one dimensional RBSDE with double barriers. Thus the wellposedness of the RBSDE is known in the literature. Several other papers have also used this tool (see *e.g.* [6, 32]). In [6], the authors consider a multi-mode switching problem. However they left open the question of the existence of the solution of the associated RBSDEs with oblique reflection. The problem is solved by Djehiche et *al.* in [10].

The paper closest to ours is a recent work by Hu and Tang [25], which we learned after we finished the first version of this paper. In their model, the generators of the RBSDEs are not interconnected and the obstacles have a special form. They take penalization approach for the existence part and establish some very nice estimates for the penalized BSDEs. Moreover, they are able to obtain the optimal strategy, as in [25], and thus uniqueness follows. However, their approach seems not working for our more general RBSDEs. In a more recent work, Hu and Tang [26] extend their results to a zero-sum game problem.

The rest of the paper is organized as follows. In next section we introduce the starting and stopping problem under Knightian uncertainty and study its relation with RBSDEs. In Section 3 we introduce the general RBSDEs and prove the existence of its solutions. We prove the uniqueness in Section 4.

2 The starting and stopping problem

In this section we introduce the starting and stopping problem under Knightian uncertainty and see how it leads to a two dimensional RBSDE with oblique reflections. We leave the general RBSDEs to next section.

2.1 The model

Let $(\Omega, \mathcal{F}, P)$ denote a fixed complete probability space on which is defined a standard d-dimensional Brownian motion $B = (B_t)_{0 \le t \le T}$, and $\mathbf{F} \triangleq (\mathcal{F}_t)_{0 \le t \le T}$ be the filtration generated by B and augmented by all the P-null sets. Throughout this paper we assume all the processes are progressively measurable and $\mathbf{F}$-adapted. Furthermore, we let:

- $\mathcal{H}$ be the space of processes η with appropriate dimensions such that $E[\int_0^T |\eta_s|^2 ds] < \infty$;

- $\mathcal{S}$ be the space of càdlàg scalar processes η such that $E[\sup_{0 \le t \le T} |\eta_t|^2] < \infty$;

- $\mathcal{S}_c$ be the subspace of $\mathcal{S}$ with continuous elements;

- $\mathcal{A}$ be the space of càdlàg and non-decreasing scalar processes η with $\eta_0 = 0$ and $E[\eta_T^2] < \infty$;

- $\mathcal{A}_c$ be the subspace of $\mathcal{A}$ with continuous elements;

Let us now fix the data of the problem.

- Let X be a process with values in $I\!R^k$ which stands for the factors determining the market electricity price.

- Let $\psi_i : [0, T] \times I\!R^k \mapsto I\!R, i = 1, 2,$ be Borelean functions which represent the utility functions for the power plant when it is in its operating and closed modes, respectively. Actually in a small interval dt, when the power plant is in its operating (resp. closed) mode it generates a profit equal to $\psi_1(t, X_t)dt$ (resp. $\psi_2(t, X_t)dt$).

- The switching of the power plant from one mode to another is not free. To be specific, if the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one at a stopping time τ, the sunk cost is equal to $\varphi_1(\tau, X_\tau)$ (resp. $\varphi_2(\tau, X_\tau)$).

We shall assume the following assumptions.

Assumption 2.1 *(i) Each component of X belongs to $\mathcal{S}_c$.*

(ii) The utility functions $\psi_i, i = 1, 2$ have linear growth in x. That is, there exists a constant C such that $|\psi_i(t, x)| \leq C(1 + |x|)$.

(iii) The cost functions $\varphi_i, i = 1, 2,$ are non-negative, jointly continuous and linearly growing in x. Moreover, $\varphi_1(t, x) + \varphi_2(t, x) > 0$ for any $(t, x) \in [0, T] \times R^k$.

(iv) The power plant is in its operating mode at the initial time $t = 0$.

We note that the above (iii) means that it is not free to make two instantaneous switching at any time $t \leq T$, and (iv) is obviously just for convenience.

Definition 2.2 *Let $\mathcal{D}$ denote the set of all admissible strategies $\delta = (\tau_n)_{n\geq 0}$ such that*

(i) τ_n's are a sequence of $\mathbf{F}$-stopping times with $\tau_0 = 0$;

(ii) $\tau_n \leq \tau_{n+1}$ for any $n \geq 0$, and $\lim_{n\to\infty} \tau_n = T$, P-a.s.

Recall Assumption 2.1 (iv), we see that τ_{2n+1} (resp. τ_{2n}) are the times where the plant is switched from the operating (resp. closed) mode to the closed (resp. operating) one.

In the conventional model, we know that the market will evolve according to the probability measure P, see e.g. [21]. Then the mean yield of the power plant when run under the strategy $\delta = (\tau_n)_{n\geq 0}$ is given by :

$$J(\delta) \triangleq E^P\Big\{ \int_0^T \psi^\delta(t, X_t)dt - A_T^\delta \Big\},$$

where E^P is the expectation under the probability measure P, and

$$\begin{cases} \psi^\delta(t, x) \triangleq \sum_{n\geq 0} \Big[\psi_1(t, x)\mathbb{1}_{[\tau_{2n}, \tau_{2n+1})}(t) + \psi_2(t, x)\mathbb{1}_{[\tau_{2n+1}, \tau_{2n+2})}(t)\Big]; \\ A_t^\delta \triangleq \sum_{n\geq 0} \Big[\varphi_1(\tau_{2n+1}, X_{\tau_{2n+1}})\mathbb{1}_{\{\tau_{2n+1}<t\}} + \varphi_2(\tau_{2n+2}, X_{\tau_{2n+2}})\mathbb{1}_{\{\tau_{2n+2}<t\}}\Big]. \end{cases} \quad (2.1)$$

Therefore the price of the power plant in the energy market is just $\sup_{\delta\in\mathcal{D}} J(\delta)$.

Knightian uncertainty amounts to suppose that we are not sure that the future will evolve under the probability P but other probabilities P^u are also likely. However we will suppose that those possible probabilities P^u are not far from P in the sense that P and P^u are equivalent. To be precise, we assume

Assumption 2.3 *(i) An uncertainty parameter u is a process taking values in some compact set U. Let $\mathcal{U}$ denote the set of all such processes u.*

(ii) Let $b : [0,T]\times\mathcal{C}([0,T],\mathbb{R}^k)\times U \mapsto \mathbb{R}^d$ be a Borel measurable and bounded function. Moreover we assume that b is continuous in u and for any $u\in\mathcal{U}$, the process $(b(t,X.,u_t))_{0\le t\le T}$ is $\mathbf{F}$-adapted.

(iii) For each $u\in\mathcal{U}$, let P^u be the probability measure define by:

$$\frac{dP^u}{dP} = L_T^u \triangleq \exp\Big(\int_0^T b(s,X.,u_s)dB_s - \frac{1}{2}\int_0^T |b(s,X.,u_s)|^2 ds\Big).$$

Since the function b is bounded, we know $\sup_{u\in\mathcal{U}} E[|L_T^u|^2]<\infty$ and P^u is indeed a probability measure. For each δ and u, denote

$$J(\delta,u) \triangleq E^u\Big\{\int_0^T \psi^\delta(t,X_t)dt - A_T^\delta\Big\}, \tag{2.2}$$

where E^u is the expectation under P^u and ψ^δ, A_T^δ are defined by (2.1). Since $X\in\mathcal{S}_c$ and ψ^δ has linear growth, we have

$$E^u\Big\{\int_0^T |\psi^\delta(t,X_t)|dt\Big\} = E\Big\{L_T^u\int_0^T |\psi^\delta(t,X_t)|dt\Big\} < \infty.$$

Here and in the sequel we denote $E\triangleq E^P$. Moreover, $A_T^\delta\ge 0$, then $J(\delta,u)$ is well defined with possible value $-\infty$. Clearly $J(\delta,u)$ is the value of the plant if the market probability turns out to be P^u and the manager follows strategy δ. However all the probability measures P^u are likely, and therefore the lowest selling price of the power plant in the energy market is given by:

$$J^* \triangleq \sup_{\delta\in\mathcal{D}} J(\delta) \quad \text{where} \quad J(\delta)\triangleq \inf_{u\in\mathcal{U}} J(\delta,u). \tag{2.3}$$

Actually the quantity J^* stands for the optimal yield of the power plant in the worst case of evolution of the market. Therefore the problem we are interested in is to asses the value J^* and, if possible, to find a pair (δ^*,u^*) such that

$$J^* = J(\delta^*) = J(\delta^*,u^*).$$

It is obvious that, for any u, $J(\delta^*, u) \geq J^*$. However, for an arbitrary δ, in general we do not have $J(\delta, u) \geq J(\delta, u^*)$.

We note that, while δ^* is obviously important in practice, the manager cannot "force" the market to evolve according to P^{u^*} because u is the uncertainty, not the manager's control. We also note that the existence of uncertainty is similar to the incompleteness in finance literature, and J^* corresponds to the sub-hedging price, or say, the minimum seller's price.

Remark 1 *In the particular case where the process X is the solution of the following standard functional SDE:*

$$dX_t = a(t, X.)dt + \sigma(t, X.)dB_t, \ t \leq T \text{ and } X_0 = x \tag{2.4}$$

with appropriate assumptions on the functions a and σ in order to guarantee existence and uniqueness of the solution of (2.4), then thanks to Girsanov's Theorem we have:

$$dX_t = (a(t, X.) + \sigma(t, X.)b(t, X., u_t))dt + \sigma(t, X.)dB_t^u, \ t \leq T \text{ and } X_0 = x$$

where $B_t^u \triangleq B_t - \int_0^t b(s, X., u_s)ds, t \leq T$, is a Brownian motion under P^u. Then our problem is equivalent to the weak formulation in stochastic control literature.

2.2 Minimization over u

In this part we fix $\delta \in \mathcal{D}$ and want to find $J(\delta)$. Recall (2.1), (2.2) and (2.3). Let $\mathcal{D}_2$ denote the space of all those $\delta \in \mathcal{D}$ such that $E[(A_T^\delta)^2] < \infty$. We first show that we need only consider $\delta \in \mathcal{D}_2$.

Lemma 2.4 *Assume Assumptions 2.1 and 2.3. Then* $\sup_{\delta \in \mathcal{D}} J(\delta) = \sup_{\delta \in \mathcal{D}_2} J(\delta)$.

Proof. Recall J^* and define $J_2^* \triangleq \sup_{\delta \in \mathcal{D}_2} J(\delta)$. Since $\mathcal{D}_2 \subset \mathcal{D}$, we have $J_2^* \leq J^*$.

To prove the other inequality, we fix an arbitrary $\delta = \{\tau_i\}_{i \geq 0} \in \mathcal{D}$. First, if $E^u\{A_T^\delta\} = \infty$ for some u, then $J(\delta) \leq J(\delta, u) = -\infty$ and thus $J(\delta) \leq J_2^*$. So without loss of generality we assume $E^u\{A_T^\delta\} < \infty$ for any u. This implies that $A_T^\delta < \infty, P^u$-a.s. Moreover, P^u is equivalent to P. Then $A_T^\delta < \infty, P$-a.s.

For any n, let $\delta^n \triangleq \{\tau_i^n\}_{i \geq 0}$, where

$$\lambda_n \triangleq \inf\{t \geq 0 : A_t^\delta \geq n\} \wedge T \quad \text{and} \quad \tau_i^n \triangleq \begin{cases} \tau_i, & \text{if } \tau_i < \lambda_n; \\ T, & \text{if } \tau_i \geq \lambda_n. \end{cases}$$

Then the stopping times $\lambda_n \uparrow T$, P-a.s., and $A^{\delta^n}_T \le n$ which implies $\delta^n \in \mathcal{D}_2$.

For any $u \in \mathcal{U}$,

$$J(\delta, u) = E^u\Big\{\int_0^T \psi^\delta(t, X_t)dt - A^\delta_T\Big\} \le E^u\Big\{\int_0^T \psi^\delta(t, X_t)dt - A^{\delta^n}_T\Big\} \triangleq J_n(\delta, u).$$

Note that

$$\begin{aligned}
&|J_n(\delta, u) - J(\delta^n, u)| \le E^u\Big\{\int_{\lambda^n}^T |\psi^\delta(t, X_t) - \psi^{\delta^n}(t, X_t)|dt\Big\}\\
&\le CE\Big\{L^u_T \int_{\lambda^n}^T (1 + |X_t|)dt\Big\} \le CE\Big\{(1 + \sup_{0\le t\le T} |X_t|)L^u_T(T - \lambda_n)\Big\}\\
&\le CE^{\frac12}\{[1 + |X_t|^2\}E^{\frac14}\{|L^u_T|^4\}E^{\frac14}\{(T - \lambda_n)^4\}.
\end{aligned}$$

Once can easily see that $\sup_{u\in\mathcal{U}} E\{|L^u_T|^4\} < \infty$. Then

$$\sup_{u\in\mathcal{U}} |J_n(\delta, u) - J(\delta^n, u)| \le CE^{\frac14}\{(T - \lambda_n)^4\} \to 0, \quad \text{as} \quad n \to \infty.$$

It follows that:

$$J(\delta, u) \le J_n(\delta, u) = J_n(\delta, u) - J(\delta^n, u) + J(\delta^n, u) \le CE^{\frac14}\{(T - \lambda_n)^4\} + J(\delta^n, u).$$

Minimize both sides over $u \in \mathcal{U}$ and recall that $\delta^n \in \mathcal{D}_2$, we get:

$$J(\delta) \le CE^{\frac14}\{(T - \lambda_n)^4\} + J(\delta^n) \le CE^{\frac14}\{(T - \lambda_n)^4\} + J^*_2.$$

Sending $n \to \infty$ we obtain the desired result. ■

We now fix $\delta \in \mathcal{D}_2$ and solve the optimization problem $J(\delta) \triangleq \inf_{u\in\mathcal{U}} J(\delta, u)$. To this end, we introduce the Hamiltonian of the problem: for any $(t, x, u, z) \in [0, T] \times \mathcal{C}([0, T], I\!R^k) \times U \times I\!R^d$,

$$H(t, x, u, z) \triangleq b(t, x, u)z \quad \text{and} \quad H^*(t, x, z) \triangleq \inf_{u\in U} H(t, x, u, z).$$

Since $b(t, x, u)$ is bounded, then the functions H and H^* are uniformly Lipschitz continuous with respect to z. Therefore, for any $u \in \mathcal{U}$, the following BSDEs have unique solutions $(Y^{\delta,u}, Z^{\delta,u}), (Y^\delta, Z^\delta) \in \mathcal{S} \times \mathcal{H}$:

$$\begin{aligned}
Y^{\delta,u}_t &= \int_t^T (\psi^\delta(s, X_s) + H(s, X_s, u_s, Z^{\delta,u}_s))ds - \int_t^T Z^{u,\delta}_s dB_s - (A^\delta_T - A^\delta_t);\\
Y^\delta_t &= \int_t^T (\psi^\delta(s, X_s) + H^*(s, X, Z^\delta_s))ds - \int_t^T Z^\delta_s dB_s - (A^\delta_T - A^\delta_t), t \le T.
\end{aligned} \tag{2.5}$$

We note that $Y^{\delta,u} + A^\delta$ and $Y^\delta + A^\delta$ are continuous. Moreover, thanks to Benes's selection Theorem [1], there exists a measurable function $u^* : [0,T] \times \mathcal{C}([0,T], I\!R^k) \times I\!R^d \mapsto U$ such that

$$H^*(t,x,z) = b(t,x,u^*(t,x,z))z.$$

Now applying Girsanov Theorem and the standard comparison theorem for BSDEs, we immediately have

Theorem 2.5 *Assume Assumptions 2.1 and 2.3, and* $\delta \in \mathcal{D}_2$.

(i) For any $u \in \mathcal{U}$, $J(\delta,u) = Y_0^{\delta,u}$.

(ii) $J(\delta) = Y_0^\delta$ *and* $u_t \stackrel{\triangle}{=} u^*(t, X_t, Z_t^\delta)$ *is optimal.*

Remark 2.6 *By using the results in Briand et al. [5], we can extend Theorem 2.5 to all* $\delta \in \mathcal{D}$ *such that* $E\{|A_T^\delta|^p\} < \infty$ *for some* $p > 1$.

2.3 RBSDEs and the verification theorem

We now maximize $J(\delta)$ over all $\delta \in \mathcal{D}_2$. Recall that we assume the initial mode is operating. For $t \in [0,\tau_1)$, we have

$$Y_t^\delta = Y_{\tau_1}^\delta - \varphi_1(\tau_1, X_{\tau_1})1\!\!1_{\{\tau_1<T\}} + \int_t^{\tau_1} \Big[\psi_1(s,X_s) + H^*(s,X_s,Z_s^\delta)\Big]ds - \int_t^{\tau_1} Z_s^\delta dB_s;$$

and for $t \in [\tau_1, \tau_2)$, we have

$$Y_t^\delta = Y_{\tau_2}^\delta - \varphi_2(\tau_2, X_{\tau_2})1\!\!1_{\{\tau_2<T\}} + \int_t^{\tau_2} \Big[\psi_2(s,X_s) + H^*(s,X_s,Z_s^\delta)\Big]ds - \int_t^{\tau_1} Z_s^\delta dB_s.$$

It is by now well known that RBSDEs is a very convenient tool to solve optimal stopping problem, see, e.g. [18]. We thus consider the following two dimensional RBSDEs:

$$\begin{cases} Y^1, Y^2 \in \mathcal{S}_c,\ Z^1, Z^2 \in \mathcal{H} \text{ and } K^1, K^2 \in \mathcal{A}_c, \\ Y_t^1 = \int_t^T \Big[\psi_1(s,X_s) + H^*(s,X_s,Z_s^1)\Big]ds - \int_t^T Z_s^1 dB_s + K_T^1 - K_t^1; \\ Y_t^2 = \int_t^T \Big[\psi_2(s,X_s) + H^*(s,X_s,Z_s^2)\Big]ds - \int_t^T Z_s^2 dB_s + K_T^2 - K_t^2; \\ Y_t^1 \geq Y_t^2 - \varphi_1(t,X_t); \quad [Y_t^1 - Y_t^2 + \varphi_1(t,X_t)]dK_t^1 = 0; \\ Y_t^2 \geq Y_t^1 - \varphi_2(t,X_t); \quad [Y_t^2 - Y_t^1 + \varphi_2(t,X_t)]dK_t^2 = 0. \end{cases} \tag{2.6}$$

Here Y_t^1 (resp. Y_t^2) stands for the optimal utility at time t if the mode at that time is operating (resp. closed). We note that, in [25] there is no Knightian uncertainty, then $H^* = 0$ and thus $\Delta Y \triangleq Y^1 - Y^2$ satisfies the following RBSDE with double reflections:

$$\begin{cases} \Delta Y_t = \int_t^T \Big[\psi_1(s, X_s) - \psi_2(s, X_s)\Big] ds - \int_t^T \Delta Z_s dB_s + (K_T^1 - K_t^1) - (K_T^2 - K_t^2); \\ -\varphi_1(t, X_t) \le \Delta Y_t \le \varphi_2(t, X_t), \ [\Delta Y_t + \varphi_1(t, X_t)] dK_t^1 = [\Delta Y_t - \varphi_2(t, X_t)] dK_t^2 = 0. \end{cases}$$

Then the wellposedness of (2.6) follows immediately.

We leave the existence of solutions of RBSDE (2.6) to next section. Our main result in this section is the following verification theorem.

Theorem 2.7 *Assume Assumptions 2.1 and 2.3, and that RBSDE (2.6) admits a solution* $(Y^i, Z^i, K^i), i = 1, 2$. *Then* $Y_0^1 = J^*$ *and the optimal strategy* $\delta^* \in \mathcal{D}_2$ *is given by* $\tau_0^* \triangleq 0$ *and, for* $n = 0, \cdots$,

$$\tau_{2n+1}^* \triangleq \inf\{t \ge \tau_{2n}^* : Y_t^1 = Y_t^2 - \varphi_1(t, X_t)\} \wedge T;$$
$$\tau_{2n+2}^* \triangleq \inf\{t \ge \tau_{2n+1}^* : Y_t^2 = Y_t^1 - \varphi_2(t, X_t)\} \wedge T.$$

Proof. (i) We first show that $Y_0^1 \ge J^*$. By Lemma 2.4 and Theorem 2.5, it suffices to show that $Y_0^1 \ge Y_0^\delta$ for any $\delta \in \mathcal{D}_2$. We thus fix an $\delta = \{\tau_n\}_{n \ge 0} \in \mathcal{D}_2$. Define

$$\bar{Z}_t^\delta \triangleq \sum_{n=0}^{\infty} \Big[Z_t^1 \mathbb{1}_{[\tau_{2n}, \tau_{2n+1})}(t) + Z_t^2 \mathbb{1}_{[\tau_{2n+1}, \tau_{2n+2})}(t) \Big], \quad 0 \le t \le T.$$

Then $\bar{Z}^\delta \in \mathcal{H}$, and

$$\begin{aligned} Y_0^1 &= Y_{\tau_1}^1 + \int_0^{\tau_1} \Big[\psi_1(s, X_s) + H^*(s, X_s, Z_s^1)\Big] ds - \int_0^{\tau_1} Z_s^1 dB_s + K_{\tau_1}^1 \\ &\ge Y_{\tau_1}^2 - \varphi_1(\tau_1, X_{\tau_1}) \mathbb{1}_{\{\tau_1 < T\}} \qquad (2.7) \\ &\quad + \int_0^{\tau_1} \Big[\psi^\delta(s, X_s) + H^*(s, X_s, \bar{Z}_s^\delta)\Big] ds + \int_0^{\tau_1} \bar{Z}_s^\delta dB_s \\ &= Y_{\tau_2}^2 + \int_{\tau_1}^{\tau_2} \Big[\psi_2(s, X_s) + H^*(s, X_s, Z_s^2)\Big] ds - \int_{\tau_1}^{\tau_2} Z_s^2 dB_s + K_{\tau_2}^2 - K_{\tau_1}^2 \\ &\quad - \varphi_1(\tau_1, X_{\tau_1}) \mathbb{1}_{\{\tau_1 < T\}} + \int_0^{\tau_1} \Big[\psi^\delta(s, X_s) + H^*(s, X_s, \bar{Z}_s^\delta)\Big] ds - \int_0^{\tau_1} \bar{Z}_s^\delta dB_s \\ &\ge Y_{\tau_2}^1 - \varphi_2(\tau_2, X_{\tau_2}) \mathbb{1}_{\{\tau_2 < T\}} - \varphi_1(\tau_1, X_{\tau_1}) \mathbb{1}_{\{\tau_1 < T\}} \qquad (2.8) \\ &\quad + \int_0^{\tau_2} \Big[\psi^\delta(s, X_s) + H^*(s, X_s, \bar{Z}_s^\delta)\Big] ds - \int_0^{\tau_2} \bar{Z}_s^\delta dB_s. \end{aligned}$$

Repeat the arguments as many times as necessary we get: for any $n \geq 0$,

$$\begin{aligned} Y_0^1 &\geq Y^1_{\tau_{2n+2}} - \sum_{k=0}^n \Big[\varphi_1(\tau_{2k+1}, X_{\tau_{2k+1}})1\!\!1_{[\tau_{2k+1}<T]} + \varphi_2(\tau_{2k+2}, X_{\tau_{2k+2}})1\!\!1_{[\tau_{2k+2}<T]}\Big] \\ &\quad + \int_0^{\tau_{2n+2}} \Big[\psi^\delta(s, X_s) + H^*(s, X, \bar Z^\delta_s)\Big]ds - \int_0^{\tau_{2n+2}} \bar Z^\delta_s dB_s \end{aligned}$$

Note that $\tau_n \uparrow T$ and Y^1 is continuous with $Y^1_T = 0$. Sending $n \to \infty$ we obtain:

$$Y_0^1 \geq \int_0^T \Big[\psi^\delta(s, X_s) + H^*(s, X, \bar Z^\delta_s)\Big]ds - \int_0^T \bar Z^\delta_s dB_s - A^\delta_T. \tag{2.9}$$

Recall (2.5). Then we have

$$\begin{aligned} Y_0^1 - Y_0^\delta &\geq \int_0^T \Big[H^*(s, X, \bar Z^\delta_s) - H^*(s, X, Z^\delta_s)\Big]ds - \int_0^T (\bar Z^\delta_s - Z^\delta_s)dB_s. \\ &= \int_0^T (\bar Z^\delta_s - Z^\delta_s)d\tilde B_s \end{aligned} \tag{2.10}$$

where $\tilde B_t \triangleq B_t - \int_0^t \gamma_s ds$ with $\gamma_s = (\gamma^1_s, \cdots, \gamma^d_s)$ and, for $i = 1, \cdots, d$,

$$\begin{aligned} \gamma^i_s &\triangleq \frac{H^*(s, X, \hat Z^{\delta,i}) - H^*(s, X, \hat Z^{\delta,i+1}_s)}{\bar Z^{\delta,i}_s - Z^{\delta,i}_s} 1\!\!1_{[\bar Z^{\delta,i}_s - Z^{\delta,i}_s \neq 0]}, \quad \text{where} \\ \hat Z^{\delta,i}_s &\triangleq (Z^{\delta,1}_s, \cdots, Z^{d,i-1}_s, \bar Z^{\delta,i}_s, \cdots, \bar Z^{\delta,d}_s). \end{aligned}$$

Since H^* is uniformly Lipschitz continuous in z, γ is bounded. Then, thanks to Girsanov's Theorem, $\tilde B$ is a Brownian motion under a new probability measure $\tilde P$ defined by:

$$\frac{d\tilde P}{dP} \triangleq \tilde L_T \triangleq \exp\Big(\int_0^T \gamma_s dB_s - \frac{1}{2}|\gamma_s|^2 ds\Big).$$

We claim that

$$E^{\tilde P}\Big\{\int_0^T (\bar Z^\delta_s - Z^\delta_s)d\tilde B_s\Big\} = 0. \tag{2.11}$$

Then by taking expectations $E^{\tilde P}$ on both sides of (2.9) we get $Y_0^1 \geq Y_0^\delta$.

The proof for (2.11) is more or less standard in BSDE literature. We nevertheless provide it for completeness. Define

$$\theta_n \triangleq \inf\{t : \int_0^t |\bar Z^\delta_s - Z^\delta_s|^2 ds \geq n\} \wedge T.$$

Then θ_n is a stopping time and $\int_0^{\theta_n} |\bar Z^\delta_s - Z^\delta_s|^2 ds \leq n$. This implies that $\int_0^{t\wedge\theta_n}(\bar Z^\delta_s - Z^\delta_s)d\tilde B_s$ is a $\tilde P$ martingale and thus

$$E^{\tilde P}\Big\{\int_0^{\theta_n} (\bar Z^\delta_s - Z^\delta_s)d\tilde B_s\Big\} = 0. \tag{2.12}$$

On the other hand, since $\bar{Z}^\delta, Z^\delta \in \mathcal{H}$ and $\tilde{P}$ and P are equivalent, we have $\int_0^T |\bar{Z}^\delta_s - Z^\delta_s|^2 ds < \infty$, $\tilde{P}$-a.s. This implies that

$$\theta_n \uparrow T, \quad \tilde{P} - a.s. \tag{2.13}$$

Moreover, for any $1 < p < 2$ and $q \triangleq \frac{2}{2-p}$ being the conjugate of $\frac{p}{2}$, applying Burkholder-Davis-Gundy inequality we have

$$\begin{aligned}
&E^{\tilde{P}}\Big\{\Big|\int_0^{\theta_n}(\bar{Z}^\delta_s - Z^\delta_s)d\tilde{B}_s\Big|^p\Big\} \le CE^{\tilde{P}}\Big\{\Big|\int_0^{\theta_n}|\bar{Z}^\delta_s - Z^\delta_s|^2 ds\Big|^{\frac{p}{2}}\Big\}\\
&= CE\Big\{\tilde{L}_T\Big|\int_0^{\theta_n}|\bar{Z}^\delta_s - Z^\delta_s|^2 ds\Big|^{\frac{p}{2}}\Big\} \le CE^{\frac{1}{q}}\{|\tilde{L}_T|^q\}E^{\frac{p}{2}}\Big\{\int_0^{\theta_n}|\bar{Z}^\delta_s - Z^\delta_s|^2 ds\Big\}\\
&\le CE^{\frac{p}{2}}\Big\{\int_0^T|\bar{Z}^\delta_s - Z^\delta_s|^2 ds\Big\} < \infty.
\end{aligned}$$

Then $\int_0^{\theta_n}(\bar{Z}^\delta_s - Z^\delta_s)d\tilde{B}_s$ are uniformly integrable under $\tilde{P}$. This, together with (2.12) and (2.13), proves (2.11).

(ii) It remains to prove $\delta^* \in \mathcal{D}_2$ and $Y^1_0 = J(\delta^*)$.

We first show that $\delta^* \in \mathcal{D}$, that is, $\tau^*_n \uparrow T, P$-a.s. Let Ω_0 denote the set of ω such that $Y^1_t(\omega), Y^2_t(\omega)$, $\varphi_1(t, X_t(\omega))$ and $\varphi_2(t, X_t(\omega))$ are continuous. Then $P(\Omega_0) = 1$. Denote $\tau^* \triangleq lim_{n\to\infty}\tau^*_n$. On $\Omega_0 \cap \{\tau^* < T\}$, for any n we have

$$\begin{aligned}
&Y^1_{\tau^*_{2n+1}}(\omega) = Y^2_{\tau^*_{2n+1}}(\omega) - \varphi_1(\tau^*_{2n+1}(\omega), X_{\tau^*_{2n+1}}(\omega)) \text{ and}\\
&Y^2_{\tau^*_{2n+2}}(\omega) = Y^1_{\tau^*_{2n+2}}(\omega) - \varphi_2(\tau^*_{2n+2}(\omega), X_{\tau^*_{2n+1}}(\omega)).
\end{aligned}$$

Send n to ∞, we obtain

$$Y^1_{\tau^*}(\omega) = Y^2_{\tau^*}(\omega) - \varphi_1(\tau^*(\omega), X_{\tau^*}(\omega)) \text{ and } Y^2_{\tau^*}(\omega) = Y^1_{\tau^*}(\omega) - \varphi_2(\tau^*(\omega), X_{\tau^*}(\omega)).$$

This obviously implies that $\varphi_1(\tau^*(\omega), X_{\tau^*}(\omega)) + \varphi_2(\tau^*(\omega), X_{\tau^*}(\omega)) = 0$, which contradicts with Assumption 2.1 (iii). Therefore $P(\tau^* = T) = 1$ and thus the strategy δ^* is admissible.

Next, by the definitions of Y^i, K^i and τ^*_1, τ^*_2 we have

$$\begin{aligned}
Y^1_{\tau^*_1} &= Y^2_{\tau^*_1} - \varphi_1(\tau^*_1, X_{\tau^*_1})1\!\!1_{\{\tau^*_1<T\}}; \quad K^1_{\tau^*_1} = 0;\\
Y^2_{\tau^*_2} &= Y^1_{\tau^*_2} - \varphi_1(\tau^*_2, X_{\tau^*_2})1\!\!1_{\{\tau^*_2<T\}}; \quad K^2_{\tau^*_2} = K^2_{\tau^*_1}.
\end{aligned}$$

Then the inequalities (2.7) and (2.8) become equalities. Repeat the arguments and since δ^* is admissible we have,

$$Y^1_0 = \int_0^T \Big[\psi^{\delta^*}(s, X_s) + H(s, X_s, \bar{Z}^{\delta^*}_s)\Big]ds - \int_0^T \bar{Z}^{\delta^*}_s dB_s - A^{\delta^*}_T. \tag{2.14}$$

That is,

$$A_T^{\delta^*} = -Y_0^1 + \int_0^T \Big[\psi^{\delta^*}(s,X_s) + H(s,X_s,\bar{Z}_s^{\delta^*})\Big]ds - \int_0^T \bar{Z}_s^{\delta^*} dB_s.$$

This implies $E[(A_T^{\delta^*})^2] < \infty$, and thus $\delta^* \in \mathcal{D}_2$. Finally, by (2.14) and Lemma 2.5 we get $Y_0^1 = J(\delta^*)$. ■

Corollary 2.8 *Assume Assumptions 2.1 and 2.3. Then the system of RBSDEs (2.6) has at most one solution.*

Proof. By Theorem 2.7 obviously Y_0^1 is unique. Similarly one can prove the uniqueness of (Y_t^1, Y_t^2). Then by Doob-Meyer Decomposition we obtain the uniqueness of Z^1, Z^2, which further implies the uniqueness of K^1 and K^2. ■

3 High Dimensional RBSDEs: Existence

In this section we extend the RBSDE (2.6) to the following general m-dimensional RBSDEs with oblique reflections for some $m \geq 2$: for $j = 1, \cdots, m$,

$$\begin{cases} Y^j \in \mathcal{S}_c,\ Z^j \in \mathcal{H} \text{ and } K^j \in \mathcal{A}_c, \\ Y_t^j = \xi_j + \int_t^T f_j(s, Y_s^1, \cdots, Y_s^m, Z_s^j)ds - \int_t^T Z_s^j dB_s + K_T^j - K_t^j; \\ Y_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i); \quad [Y_t^j - \max_{i \in A_j} h_{j,i}(t, Y_t^i)]dK_t^j = 0. \end{cases} \tag{3.1}$$

Here ξ_j are $\mathcal{F}_T$-measurable, the coefficients $f_j, h_{j,i}$ can depend on ω, and $A_j \subset \{1, \cdots, m\} - \{j\}$. For simplicity we denote $\overrightarrow{Y_t} \triangleq (Y_t^1, \cdots, Y_t^m)$, and similarly for other vectors. We emphasize that here A_j can be empty and if so we take the convention that the maximum over the empty set, denoted as $\emptyset$, is $-\infty$. Then in this case Y^j has no lower barrier and then we take $K^j = 0$. Consequently, Y^j satisfies the following BSDE without reflection:

$$Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)ds - \int_t^T Z_s^j dB_s,\ 0 \leq t \leq T.$$

Also, for any j we define

$$h_{j,j}(t,y) \triangleq y. \tag{3.2}$$

Then a solution of (3.1) always satisfies

$$Y_t^j \geq \max_{i \in A_j \cup \{j\}} h_{j,i}(t, Y_t^i). \tag{3.3}$$

The motivation of studying multi-dimensional RBSDEs is of course the multi-mode switching problems. The constraint A_j means that from mode j the plant can only be switched to those modes in A_j. The general barrier $h_{j,i}$ allows one to consider more general switching cost, and the dependence of f_j on Y^j can be interpreted as recursive utilities, see, e.g. Duffie-Epstein [15] and [16]. Our model is even more general in the sense that f_j may depend on other Y^i as well. This can be potentially useful in game problems, where the utilities of the m players are recursive and interconnected. We note that the monotonicity assumption in Assumption 3.1 (iii) implies that the m players are "partners" with positively correlated interests.

We note that recently Hu and Tang [25] also studied RBSDEs with oblique reflections. In their model, the generators take the form $f_j(t, Y_t^j, Z_t^j)$ and the barriers take the form $h_{j,i}(t, Y_t^i) = Y_t^i - c_{j,i}$. They take penalization approach for the existence part and establish some very nice estimates for the penalized BSDEs. For the uniqueness, they proved a verification theorem and obtain the optimal strategy, in the spirit of Theorem 2.7. However, it seems that their approaches do not work for our more general RBSDEs (3.1).

We will leave the more tricky uniqueness issue to next section. To prove the existence of solutions, we use the notion of the smallest g-supermartingales introduced by Peng [29] and Peng and Xu [30], which can be understood as a nonlinear version of the snell envelope.

Throughout this section we shall adopt the following assumptions.

Assumption 3.1 *For any $j = 1, \cdots, m$, it holds that:*

(i) $E\Big\{\int_0^T \sup_{\overrightarrow{y}: y_j = 0} |f_j(t, \overrightarrow{y}, 0)|^2 dt + |\xi_j|^2\Big\} < \infty.$

(ii) $f_j(t, \overrightarrow{y}, z)$ is uniformly Lipschitz continuous in (y_j, z) and is continuous in y_i for any $i \neq j$; and $h_{j,i}(t, y)$ is continuous in (t, y) for $i \in A_j$,.

(iii) $f_j(t, \overrightarrow{y}, z)$ is increasing in y_i for $i \neq j$, and $h_{j,i}(t, y)$ is increasing in y for $i \in A_j$.

(iv) For $i \in A_j$, $h_{j,i}(t, y) \leq y$. Moreover, there is no sequence $j_2 \in A_{j_1}, \cdots, j_k \in$

$A_{j_{k-1}}, j_1 \in A_{j_k}$, *and* $(y_1, \cdots, y_k)$ *such that*

$$y_1 \triangleq h_{j_1,j_2}(t, y_2), y_2 \triangleq h_{j_2,j_3}(t, y_3), \cdots, y_{k-1} \triangleq h_{j_{k-1},j_k}(t, y_k), y_k \triangleq h_{j_k,j_1}(t, y_1).$$

(v) For any $j = 1, ..., m$, $\xi_j \geq \max_{i \in A_j} h_{j,i}(T, \xi_i)$. ■

We note that (i), (ii) and (v) are standard; and (iii) implies the m players are "partners". The assumption (iv) means that it is not free to make a circle of instantaneous switchings. This is satisfied if, for example, $h_{j,i}(\omega, t, y) = y - \varphi_{j,i}(\omega, t)$ with $\varphi_{j,i}(\omega, t) > 0$.

Our main result of this section is:

Theorem 3.2 *Assume Assumption 3.1 holds. Then RBSDE (3.1) has a solution.*

Note that Assumptions 2.1 and 2.3 obviously imply Assumption 3.1. Then combining Theorem 3.2 and Corollary 2.8 we immediately have

Corollary 3.3 *Assume Assumptions 2.1 and 2.3 hold. Then RBSDE (2.6) has a unique solution.*

Proof of Theorem 3.2. We shall use Picard iteration, and proceed several steps.

Step 1. We first construct the Picard iterations. Denote:

$$\underline{f}_j(t, y, z) \triangleq \inf_{\overrightarrow{y}: y_j = y} f_j(t, \overrightarrow{y}, z) \quad \text{and} \quad \bar{f}_j(t, y, z) \triangleq \sup_{\overrightarrow{y}: y_j = y} f_j(t, \overrightarrow{y}, z).$$

By Assumption 3.1 (i) and (ii), $\underline{f}_j, \bar{f}_j$ are uniformly Lipschitz continuous in (y, z) and

$$E\Big\{ \int_0^T [|\underline{f}_j(t, 0, 0)|^2 + |\bar{f}_j(t, 0, 0)|^2] dt \Big\} < \infty.$$

Let $(Y^{j,0}, Z^{j,0})$ be the solution to the following BSDE without reflection:

$$Y_t^{j,0} = \xi_j + \int_t^T \underline{f}_j(s, Y_s^{j,0}, Z_s^{j,0}) ds - \int_t^T Z_s^{j,0} dB_s, \quad j = 1, \cdots, m. \tag{3.4}$$

For $j = 1, \ldots, m$ and $n = 1, 2, \cdots$, recursively define $Y^{j,n}$ via the following RBSDEs whose solution exits thanks to the result by El-Karoui et al. [18]:

$$\begin{cases} Y_t^{j,n} = \xi_j - \int_t^T Z_s^{j,n} dB_s + K_T^{j,n} - K_t^{j,n} \\ \quad + \int_t^T f_j(s, Y_s^{1,n-1}, \cdots, Y_s^{j-1,n-1}, Y_s^{j,n}, Y_s^{j+1,n-1} \cdots, Y_s^{m,n-1}, Z_s^{j,n}) ds; \\ Y_t^{j,n} \geq \max_{i \in A_j} h_{j,i}(t, Y_t^{i,n-1}); \quad [Y_t^{j,n} - \max_{i \in A_j} h_{j,i}(t, Y_t^{i,n-1})] dK_t^{j,n} = 0. \end{cases} \tag{3.5}$$

Note that, given $Y^{i,n-1}, i = 1, \cdots, m$, for each j (3.5) is a one dimensional BSDE (when $A_j = \phi$) or RBSDE. Under Assumption 3.1, (3.5) has a unique solution. Moreover, by comparison theorem (see e.g. [18], Theorem 4.1) it is obvious that $Y^{j,1} \geq Y^{j,0}$. Then by induction one can easily show that $Y^{j,n}$ is increasing as n increases.

Step 2. We show that

$$E\Big\{ \sup_{0\leq t\leq T} |Y_t^{j,n}|^2 + \int_0^T |Z_t^{j,n}|^2 dt + |K_T^{j,n}|^2 \Big\} \leq C, \quad \forall j, n. \tag{3.6}$$

To this end, denote:

$$\breve{\xi} \triangleq \sum_{j=1}^m |\xi_j| \quad \text{and} \quad \breve{f}(t,y,z) \triangleq \sum_{j=1}^m |\bar{f}_j(t,y,z)|,$$

and let $(\breve{Y}, \breve{Z})$ be the solution to the following BSDE:

$$\breve{Y}_t = \breve{\xi} + \int_t^T \breve{f}(s, \breve{Y}_s, \breve{Z}_s) ds - \int_t^T \breve{Z}_s dB_s.$$

Denote, for $j = 1, \cdots, m$,

$$\bar{Y}_t^j \triangleq \breve{Y}_t, \quad \bar{Z}_t^j \triangleq \breve{Z}_t, \quad \bar{K}_t^j \triangleq 0.$$

Obviously $Y_t^{j,0} \leq \bar{Y}_t^j$. Note that $(\bar{Y}^j, \bar{Z}^j, \bar{K}^j)$ satisfies

$$\begin{cases} \bar{Y}_t^j = \breve{\xi} + \int_t^T \breve{f}(s, \bar{Y}_s^j, \bar{Z}_s^j) - \int_t^T \bar{Z}_s^j dB_s + \bar{K}_T^j - \bar{K}_t^j; \\ \bar{Y}_t^j \geq \max_{i\in A_j} h_{j,i}(t, \bar{Y}_t^i); \quad [\bar{Y}_t^j - \max_{i\in A_j} h_{j,i}(t, \bar{Y}_t^i)] d\bar{K}_t^j = 0. \end{cases}$$

Once more apply the comparison theorem repeatedly, we get

$$Y_t^{j,n} \leq \breve{Y}_t, \quad \forall n.$$

Recall that $Y_t^{j,n} \geq Y_t^{j,0}$. Then

$$\sum_{j=1}^m E\Big\{ \sup_{0\leq t\leq T} |Y_t^{j,n}|^2 \Big\} \leq C < \infty, \quad \forall n. \tag{3.7}$$

Moreover,

$$E\Big\{ \sup_{0\leq t\leq T} |[\max_{i\in A_j} h_{j,i}(t, Y_t^{i,n-1})]^+|^2 \Big\} \leq E\Big\{ \sup_{0\leq t\leq T} |[\max_{i\in A_j} Y_t^{i,n-1}]^+|^2 \Big\} \leq C.$$

This, together with (3.7) and applying the results in [18], proves (3.6).

Step 3. Now let Y^j denote the limit of $Y^{j,n}$. By Peng's monotonic limit theorem [29] or [30], we know Y^j is an càdlàg process, and following similar arguments there one can easily show that there exist $Z^j \in \mathcal{H}$ and $K^j \in \mathcal{A}$ such that

$$\begin{cases} Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)ds - \int_t^T Z_s^j dB_s + K_T^j - K_t^j; \\ Y_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i). \end{cases} \tag{3.8}$$

Consider now the following RBSDEs whose solution exits thanks to the result by Hamadène [20] or Peng and Xu [30]:

$$\begin{cases} \tilde{Y}^j \in \mathcal{S}, \quad \tilde{Z}^j \in \mathcal{H} \quad \text{and} \quad \tilde{K}^j \in \mathcal{A}; \\ \tilde{Y}_t^j = \xi_j - \int_t^T \tilde{Z}_s^j dB_s + \tilde{K}_T^j - \tilde{K}_t^j \\ \qquad\qquad + \int_t^T f_j(s, Y_s^1, \cdots, Y_s^{j-1}, \tilde{Y}_s^j, Y_s^{j+1}, \cdots, Y_s^m, \tilde{Z}_s^j)ds; \\ \tilde{Y}_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i); \quad [\tilde{Y}_{t-}^j - \max_{i \in A_j} h_{j,i}(t, Y_{t-}^i)]d\tilde{K}_t^j = 0. \end{cases} \tag{3.9}$$

We note that (3.8) and (3.9) have the same lower barrier. Since $\tilde{Y}^j$ is the smallest f_j-supermartingale with lower barrier $\max_{i \in A_j} h_{j,i}(t, Y_t^i)$, we have $\tilde{Y}_t^j \leq Y_t^j$ (see [30], Theorem 2.1). On the other hand, since $Y_t^{i,n-1} \leq Y_t^i$ for any $(i, n-1)$, by the monotonicity of $h_{j,i}$ we get

$$\max_{i \in A_j} h_{j,i}(t, Y_t^{i,n-1}) \leq \max_{i \in A_j} h_{j,i}(t, Y_t^i).$$

Then once more by comparison theorem for RBSDEs we have $Y_t^{j,n} \leq \tilde{Y}_t^j$, which implies that $Y_t^j \leq \tilde{Y}_t^j$. Therefore, $\tilde{Y}_t^j = Y_t^j$. This further implies that $\tilde{Z}_t^j = Z_t^j$, $dt \otimes dP$-a.s., $\tilde{K}_t^j = K_t^j$ for any $0 \leq t \leq T, P$-a.s., and that

$$\begin{cases} Y_t^j = \xi_j + \int_t^T f_j(s, \overrightarrow{Y}_s, Z_s^j)ds - \int_t^T Z_s^j dB_s + K_T^j - K_t^j; \\ Y_t^j \geq \max_{i \in A_j} h_{j,i}(t, Y_t^i), \quad [Y_{t-}^j - \max_{i \in A_j} h_{j,i}(t, Y_{t-}^i)]dK_t^j = 0.. \end{cases} \tag{3.10}$$

Step 4. We show that Y^j is continuous. This obviously implies that K^j is also continuous and thus $(\overrightarrow{Y}, \overrightarrow{Z}, \overrightarrow{K})$ is a solution to (3.1).

We first note that, by (3.10), $\Delta Y_t^j = -\Delta K_t^j \leq 0$, and if $\Delta K_t^j \neq 0$, then $Y_{t-}^j = \max_{i \in A_j} h_{j,i}(t, Y_{t-}^i)$. It is obvious that Y^j is continuous when $A_j = \emptyset$. We now assume

$\Delta Y_t^{j_1} \neq 0$ for some j_1 and t. Then $A_{j_1} \neq \emptyset$ and $\Delta Y_t^{j_1} < 0$. Note that in this case $\Delta K_t^{j_1} > 0$, which further implies that

$$Y_{t-}^{j_1} = \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_{t-}^i).$$

Let $j_2 \in A_{j_1}$ be the optimal index, then

$$h_{j_1,j_2}(t, Y_{t-}^{j_2}) = Y_{t-}^{j_1} > Y_t^{j_1} \geq \max_{i \in A_{j_1}} h_{j_1,i}(t, Y_t^i) \geq h_{j_1,j_2}(t, Y_t^{j_2}).$$

Thus $\Delta Y_t^{j_2} < 0$, and therefore $A_{j_2} \neq \emptyset$. Repeat the arguments we obtain $j_k \in A_{j_{k-1}}$ and $\Delta Y_t^{j_k} < 0$ for any k. Since each j_k can take only values $1, \cdots, m$, we may assume, without loss of generality that $j_1 = j_{k+1}$ for some $k \geq 2$ (note again that $j_1 \notin A_{j_1}$ and thus $j_2 \neq j_1$). Then we have

$$Y_{t-}^{j_1} = h_{j_1,j_2}(t, Y_{t-}^{j_2}), \cdots, Y_{t-}^{j_{k-1}} = h_{j_{k-1},j_k}(t, Y_{t-}^{j_k}), \quad Y_{t-}^{j_k} = h_{j_k,j_1}(t, Y_{t-}^{j_1}).$$

This contradicts with Assumption 3.1 (iv). Therefore, all processes Y^j are continuous.

Step 5. Finally, as a by product we show that, for $j = 1, \cdots, m$,

$$\lim_{n\to\infty} E\Big\{ \sup_{0 \leq t \leq T} [|Y_t^{j,n} - Y_t^j|^2 + |K_t^{j,n} - K_t^j|^2] + \int_0^T |Z_t^{j,n} - Z_t^j|^2 dt \Big\} = 0. \tag{3.11}$$

In fact, since Y^j is continuous and $Y^{j,n} \uparrow Y^j$, by Dini's Theorem we know

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} |Y_t^{j,n} - Y_t^j| = 0, \quad a.s.$$

Applying Dominated Convergence Theorem we prove the convergence of $Y^{j,n}$ in (3.11). Now by standard arguments, see e.g. [18], one can prove (3.11). ■

By applying comparison theorem repeatedly, the following two results are direct consequences of Theorem 3.2, and their proofs are omitted.

Corollary 3.4 *The solution $\overrightarrow{Y}$ constructed in Theorem 3.2 is the minimum solution of (3.1). That is, if $\overrightarrow{\tilde{Y}}$ is another solution of (3.1), then $Y_t^j \leq \tilde{Y}_t^j, j = 1, \cdots, m$.*

Corollary 3.5 *Assume $(\tilde{\xi}_j, \tilde{f}_j)$ also satisfy Assumption 3.1, and*

$$f_j \leq \tilde{f}_j, \quad \xi_j \leq \tilde{\xi}_j.$$

Let $\overrightarrow{Y}$ and $\overrightarrow{\tilde{Y}}$ denote the solution of (3.1) constructed in Theorem 3.2, with coefficients $(\xi_j, f_j, h_{j,i})$ and $(\tilde{\xi}_j, \tilde{f}_j, h_{j,i})$, respectively. Then $Y_t^j \leq \tilde{Y}_t^j, j = 1, \cdots, m$.

We also have the convergence of the penalized BSDEs, which is obtained by Hu and Tang [25] in their case using a different approach.

Theorem 3.6 *Assume Assumption 3.1 holds, and* $(\overrightarrow{Y}, \overrightarrow{Z}, \overrightarrow{K})$ *denote the solution of (3.1) constructed in Theorem 3.2. Let* $(\overrightarrow{Y^n}, \overrightarrow{Z^n})$ *denote the solutions of the following penalized BSDEs without reflection:*

$$Y_t^{n,j} = \xi_j + \int_t^T f_j(s, \overrightarrow{Y_s^n}, Z_s^{n,j})ds + n\int_t^T [Y_s^{n,j} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i})]^- ds - \int_t^T Z_s^{n,j} dB_s. \tag{3.12}$$

Then $Y^{n,j}$ *is increasing in* n *and*

$$\lim_{n\to\infty} E\Big\{ \sup_{0\le t\le T} [|Y_t^{n,j} - Y_t^j|^2 + |K_t^{n,j} - K_t^j|^2] + \int_0^T |Z_t^{n,j} - Z_t^j|^2 dt \Big\} = 0, \tag{3.13}$$

where

$$K_t^{n,j} \triangleq n\int_0^t [Y_s^{n,j} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i})]^- ds.$$

Proof. The proof is similar to Theorem 3.2, we thus only introduce the main idea and leave the details to the interested readers.

First, it is obvious that the BSDEs (3.12) have a unique solution. Define $Y_t^{n,j,0} \triangleq Y_t^{j,0}$, and for $k = 0, 1, \cdots$, recursively define

$$\begin{aligned} Y_t^{n,j,k+1} &= \xi_j + \int_t^T f_j(s, Y_s^{n,1,k}, \cdots, Y_s^{n,j-1,k}, Y_s^{n,j,k+1}, Y_s^{n,j+1,k}, \cdots, Y_s^{n,m,k}, Z_s^{n,j})ds \\ &\quad + n\int_t^T [Y_s^{n,j,k+1} - \max_{i\in A_j} h_{j,i}(s, Y_s^{n,i,k})]^- ds - \int_t^T Z_s^{n,j,k+1} dB_s. \end{aligned}$$

By standard arguments in BSDE theory one can easily see that

$$\lim_{k\to\infty} E\Big\{ \sup_{0\le t\le T} |Y_t^{n,j,k} - Y_t^{n,j}|^2 + \int_0^T |Z_t^{n,j,k} - Z_t^{n,j}|^2 \Big\} = 0.$$

Moreover, by comparison theorem we have $Y^{n,j,k}$ is increasing in n. Thus $Y^{n,j}$ is increasing in n. We note that one can also use the comparison theorem for high dimensional BSDEs, see Hu and Peng [24], to prove the monotonicity of $Y^{n,j}$. Let $\tilde{Y}^j$ denote the limit of $Y^{n,j}$ as $n \to \infty$. By induction one can show that $Y^{n,j,k} \le Y^{j,k}$ for any (n, j, k). Then $Y^{n,j} \le Y^j$ and thus $\tilde{Y}^j \le Y^j$. Now apply the results in [30] and the arguments in Theorem 3.2, we can prove $\tilde{Y}^j = Y^j$ and (3.13). ∎

Another by-product of Theorem 3.2 is the existence of a solution of the system (3.1) considered between two stopping times. This result is in particular useful to show uniqueness of (3.1) in next section.

To be precise, let λ_1 and λ_2 be two stopping times such that P-$a.s.$, $0 \le \lambda_1 \le \lambda_2 \le T$ and let us consider the following RBSDE over $[\lambda_1, \lambda_2]$: for $j = 1, \cdots, m$, P-$a.s.$,

$$\begin{cases} (Y_t^j)_{t\in[\lambda_1,\lambda_2]} \text{ continuous, } (K_t^j)_{t\in[\lambda_1,\lambda_2]} \text{ continuous and nondecreasing,} \\ K_{\lambda_1}^j = 0, \text{ and } E\Big\{\sup_{t\in[\lambda_1,\lambda_2]} |Y_t^j|^2 + \int_{\lambda_1}^{\lambda_2} |Z_s^j|^2 ds + (K_{\lambda_2}^j)^2\Big\} < \infty; \\ Y_t^j = \xi_{\lambda_2}^j + \int_t^{\lambda_2} f_j(s, \overrightarrow{Y}_s, Z_s^j) ds - \int_t^{\lambda_2} Z_s^j dB_s + K_{\lambda_2}^j - K_t^j, \ \forall t \in [\lambda_1, \lambda_2]\,; \\ Y_t^j \ge \max_{i\in A_j} h_{j,i}(t, Y_t^i) \text{ and } [Y_t^j - \max_{i\in A_j} h_{j,i}(t, Y_t^i)] dK_t^j = 0, \ \forall t \in [\lambda_1, \lambda_2]. \end{cases} \tag{3.14}$$

Then we have:

Theorem 3.7 *: Assume Assumption 3.1 holds and that, for $j = 1, ..., m$, $\xi_{\lambda_2}^j \in \mathcal{F}_{\lambda_2}$ and satisfies:*

$$E\{|\xi_{\lambda_2}^j|^2\} < \infty \text{ and } \xi_{\lambda_2}^j \ge \max_{i\in A_j} h_{j,i}(\lambda_2, \xi_{\lambda_2}^i). \tag{3.15}$$

Then the RBSDE (3.14) has a solution. ■

4 High Dimensional RBSDEs: Uniqueness

We now focus on uniqueness of the solution of RBSDE (3.14), hence that of RBSDE (3.1). To do that we need a stronger assumption.

Assumption 4.1 *(i) f_j is uniformly Lipschitz continuous in all y_i.*

(ii) If $i \in A_j, k \in A_i$, then $k \in A_j \cup \{j\}$. Moreover,

$$h_{j,i}(t, h_{i,k}(t, y)) < h_{j,k}(t, y). \tag{4.1}$$

(iii) For any $i \in A_j$,

$$|h_{j,i}(t, y_1) - h_{j,i}(t, y_2)| \le |y_1 - y_2|. \tag{4.2}$$

Note that these assumptions are satisfied if $A_j = \{1, \dots, m\} - \{j\}$ for any $j = 1, ..., m$ and $h_{j,i}(\omega, t, y) = y - \varphi_{j,i}(\omega, t)$ with $\varphi_{j,i}(\omega, t) > 0$ for any $t \le T$, P-$a.s.$

Our main result of this section is the following theorem.

Theorem 4.2 (Uniqueness)

(i) Assume Assumptions 3.1 and 4.1 are in force, and $\xi^j_{\lambda_2}$ *satisfies (3.15). Then the solution of RBSDE (3.14) is unique.*

(ii) Moreover, assume for $j = 1, \dots, m$, $\tilde{f}_j$ *satisfies Assumptions 3.1 and 4.1, and* $\tilde{\xi}^j_{\lambda_2}$ *satisfies (3.15). Let* $(\tilde{Y}^j, \tilde{Z}^j)$ *be the solution to RBSDE (3.14) corresponding to* $(\tilde{f}_j, \tilde{\xi}^j_{\lambda_2})$. *For* $j = 1, ..., m$, *denote,*

$$\Delta Y^j_t \triangleq Y^j_t - \tilde{Y}^j_t, \quad \Delta \xi^j_{\lambda_2} \triangleq \xi^j_{\lambda_2} - \tilde{\xi}^j_{\lambda_2}, \quad \|\Delta f_t\| \triangleq \sum_{j=1}^m \operatorname*{esssup}_{(\overrightarrow{y}, z)} |[f_j - \tilde{f}_j](t, \overrightarrow{y}, z)|. \quad (4.3)$$

Then there exists a constant C, *which is independent of* λ_1, λ_2, *such that:*

$$\max_{1 \le j \le m} |\Delta Y^j_{\lambda_1}|^2 \le E_{\lambda_1}\Big\{ e^{C(\lambda_2 - \lambda_1)} \max_{1 \le j \le m} |\Delta \xi^j_{\lambda_2}|^2 + C \int_{\lambda_1}^{\lambda_2} \|\Delta f_t\|^2 dt \Big\}. \quad (4.4)$$

We note that the stability result (4.4) is not only interesting in its own right, we need it to prove the uniqueness in (i).

The main idea is to prove a verification theorem in the spirit of Theorem 2.7. However, the proof here is much more involved because for our general RBSDEs the optimal strategy like the δ^* in Theorem 2.7 does not exist. We can only construct some approximately optimal strategy, and then we need some precise estimates of the errors, which will be obtained by using (4.4).

The rest of this section is organized as follows. In Section 4.1 we discuss heuristically how to find the approximately optimal strategies, which will lead to the definition of admissible strategies. In Section 4.2 we define rigorously the admissible strategy δ and the corresponding value function Y^δ. In Section 4.3 we estimate the error between $Y^{\delta,j}$ and the given solution Y^j, which leads to the verification theorem. Finally in Section 4.4 we prove Theorem 4.2.

4.1 Heuristic discussion

We want to extend the arguments in Theorem 2.7 to this case. For an arbitrary solution, the idea is to express Y^1_0 as the supremum of Y^δ_0 for some appropriately defined Y^δ. The strategy δ we can use here is more subtle. To explain the difference and to motivate our definition of admissible strategies, let us first consider the following two dimensional RBSDEs:

$$\begin{cases} Y_t^j = \xi_j + \int_t^T f_j(s, Y_s^1, Y_s^2, Z_s^j)ds - \int_t^T Z_s^j dB_s + K_T^j - K_t^j, \quad j = 1,2; \\ Y_t^1 \geq h_1(t, Y_t^2); \quad [Y_t^1 - h_1(t, Y_t^2)]dK_t^1 = 0; \\ Y_t^2 \geq h_2(t, Y_t^1); \quad [Y_t^2 - h_2(t, Y_t^1)]dK_t^2 = 0. \end{cases} \tag{4.5}$$

Assume (Y^1, Y^2) is an arbitrary solution of (4.5). As in Theorem 2.7 we want to express Y_0^1 as $Y_0^{\delta^*}$ for some δ^* and appropriately defined Y^{δ^*}. Very naturally we want to define

$$\tau_1^* \triangleq \inf\{t \geq 0 : Y_t^1 = h_1(t, Y_t^2)\} \wedge T. \tag{4.6}$$

When f_1 does not depend on Y_t^2, as in (2.6) or in Hu and Tang [25], we have

$$Y_t^1 = \xi_1 1\!\!1_{\{\tau_1^*=T\}} + h_1(\tau_1^*, Y_{\tau_1^*}^2) 1\!\!1_{\{\tau_1^*<T\}} + \int_t^{\tau_1^*} f_1(s, Y_s^1, Z_s^1)ds - \int_t^{\tau_1^*} Z_s^1 dB_s.$$

This is a BSDE without reflection and is wellposed. Therefore, once we can determine $Y_{\tau_1^*}^2$, Y_t^1 is unique on $[0, \tau_1^*]$. Next we can define τ_2^* by using Y^2 and express $Y_{\tau_1^*}^2$ in terms of $Y_{\tau_2^*}^1$. Repeat the arguments we can mimic the proof of Theorem 2.7.

However, in our case, we have to consider the following RBSDE over $[0, \tau_1^*]$,

$$\begin{cases} Y_t^1 = \xi_1 1\!\!1_{\{\tau_1^*=T\}} + h_1(\tau_1^*, Y_{\tau_1^*}^2) 1\!\!1_{\{\tau_1^*<T\}} + \int_t^{\tau_1^*} f_1(s, Y_s^1, Y_s^2, Z_s^1)ds - \int_t^{\tau_1^*} Z_s^1 dB_s; \\ Y_t^2 = Y_{\tau_1^*}^2 + \int_t^{\tau_1^*} f_2(s, Y_s^1, Y_s^2, Z_s^2)ds - \int_t^{\tau_1^*} Z_s^2 dB_s + K_T^2 - K_t^2; \\ Y_t^2 \geq h_2(t, Y_t^1); \quad [Y_t^2 - h_2(t, Y_t^1)]dK_t^2 = 0. \end{cases} \tag{4.7}$$

This itself takes the form of (3.14), whose wellposedness needs to be proved. We will come back to this idea later.

There is another naive approach. Define

$$\tau_1^* \triangleq \inf\{t > 0 : Y_t^1 = h_1(t, Y_t^2) \ \text{ or } \ Y_t^2 = h_2(t, Y_t^1)\} \wedge T$$

Then we have

$$\begin{cases} Y_t^1 = Y_{\tau_1^*}^1 + \int_t^{\tau_1^*} f_1(s, Y_s^1, Y_s^2, Z_s^1)ds - \int_t^{\tau_1^*} Z_s^1 dB_s; \\ Y_t^2 = Y_{\tau_1^*}^2 + \int_t^{\tau_1^*} f_2(s, Y_s^1, Y_s^2, Z_s^2)ds - \int_t^{\tau_1^*} Z_s^2 dB_s; \end{cases} \quad 0 \leq t \leq \tau_1^*.$$

This system is wellposed once the terminal conditions are given. However, in this approach we will have to define

$$\tau_2^* \triangleq \inf\{t > \tau_1^* : Y_t^1 = h_1(t, Y_t^2) \ \text{ or } \ Y_t^2 = h_2(t, Y_t^1)\} \wedge T.$$

In this case it is likely that $\tau_2^* = \tau_1^*$, and then we have trouble to move forward.

We now come back to the first approach. That is, we consider (4.6) and (4.7). One key observation is that, although we do not know its uniqueness yet, RBSDE (4.7) has only one reflection while the original RBSDE (4.5) has two reflections. Therefore, by doing this we reduce the number of reflections, and thus by repeating the procedure we can transform the system to BSDEs without reflection which is wellposed.

There is another difficulty to prove the verification theorem for RBSDEs in the form of (4.7). To illustrate the idea let us consider the following RBSDE instead of (4.7):

$$\begin{cases} Y_t^1 = \xi_1 + \int_t^T f_1(s, Y_s^1, Y_s^2, Z_s^1)ds - \int_t^T Z_s^1 dB_s + K_T^1 - K_t^1; \\ Y_t^2 = \xi_2 + \int_t^T f_2(s, Y_s^1, Y_s^2, Z_s^2)ds - \int_t^T Z_s^2 dB_s; \\ Y_t^1 \ge h_1(t, Y_t^2); \quad [Y_t^1 - h_1(t, Y_t^2)]dK_t^1 = 0. \end{cases} \tag{4.8}$$

Again we define τ_1^* by (4.6). Then over $[0, \tau_1^*]$ we have

$$\begin{cases} Y_t^1 = \xi_1 \mathbb{1}_{\{\tau_1^* = T\}} + h_1(\tau_1^*, Y_{\tau_1^*}^2)\mathbb{1}_{\{\tau_1^* < T\}} + \int_t^{\tau_1^*} f_1(s, Y_s^1, Y_s^2, Z_s^1)ds - \int_t^{\tau_1^*} Z_s^1 dB_s; \\ Y_t^2 = Y_{\tau_1^*}^2 + \int_t^{\tau_1^*} f_2(s, Y_s^1, Y_s^2, Z_s^2)ds - \int_t^{\tau_1^*} Z_s^2 dB_s. \end{cases}$$

This is wellposed. However, Y^2 has no reflection, thus we cannot define τ_2^* as in Theorem 2.7. Our second key observation is that, when $\tau_1^* < T$, $Y_{\tau_1^*}^1 = h_1(\tau_1^*, Y_{\tau_1^*}^2)$. Note that Y^1, Y^2, h are all continuous. This implies that if τ_2^* is close to τ_1^*, then $Y_t^1 \approx h_1(t, Y_t^2)$ for $t \in [\tau_1^*, \tau_2^*]$, and therefore,

$$Y_t^2 \approx Y_{\tau_2^*}^2 + \int_t^{\tau_2^*} f_2(s, h_1(t, Y_t^2), Y_s^2, Z_s^2)ds - \int_t^{\tau_2^*} Z_s^2 dB_s. \tag{4.9}$$

Ignoring the approximation, this is a BSDE without reflection and is wellposed. We should, of course, estimate the error due to this approximation.

We now summarize the above idea and discuss heuristically how to find the approximately optimal strategy for the m dimensional RBSDE (3.14). Let μ denote the number of nonempty sets A_j in (3.14), that is, the number of reflections in (3.14). We proceed by induction on μ. First, when $\mu = 0$, (3.14) becomes an m-dimensional BSDE without reflection. By standard arguments one can easily show that Theorem 4.2 holds. Now assume Theorem 4.2 is true for $\mu = m_1 - 1$ for some $1 \le m_1 \le m$. For $\mu = m_1$, let (Y^j, Z^j, K^j) be an arbitrary solution of (3.14).

Let $\tau_0^* \stackrel{\triangle}{=} \lambda_1$, and without loss of generality assume $A_1 \neq \emptyset$. Set

$$\tau_1^* \stackrel{\triangle}{=} \inf\{t \geq \tau_0^* : Y_t^1 = \max_{i\in A_1} h_{1,i}(t, Y_t^i)\} \wedge \lambda_2.$$

When $\tau_1^* < \lambda_2$, we have

$$Y_{\tau_1^*}^1 = \max_{i\in A_1} h_{1,i}(\tau_1^*, Y_{\tau_1^*}^i).$$

That is, there exists an index, denoted as $\eta_1 \in A_1$, such that

$$Y_{\tau_1^*}^1 = h_{1,\eta_1}(\tau_1^*, Y_{\tau_1^*}^{\eta_1}).$$

So, besides the stopping time τ_1^*, we need to keep track of the *optimal index* η_1. We note that η_1 is random and is $\mathcal{F}_{\tau_1^*}$ measurable. At this point, let us denote $\eta_0 \stackrel{\triangle}{=} 1$. Note that, over $[\tau_0^*, \tau_1^*]$, it holds that:

$$\begin{cases} Y_t^j = Y_{\tau_1^*}^j + \int_t^{\tau_1^*} f_j(s, \overrightarrow{Y}_s, Z_s^j)ds - \int_t^{\tau_1^*} Z_s^j dB_s + K_{\tau_1^*}^j - K_t^j, \ j \neq \eta_0; \\ Y_t^j \geq \max_{k\in A_j} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k\in A_j} h_{j,k}(t, Y_t^k)]dK_t^j = 0, \ j \neq \eta_0; \\ Y_t^{\eta_0} = Y_{\tau_1^*}^{\eta_0} + \int_t^{\tau_1^*} f_{\eta_0}(s, \overrightarrow{Y}_s, Z_s^{\eta_0})ds - \int_t^{\tau_1^*} Z_s^{\eta_0} dB_s. \end{cases} \tag{4.10}$$

This is a system with only $m_1 - 1$ reflections, and thus is wellposed by our induction assumption.

Now assume $\tau_1^* < \lambda_2$. To define (τ_2^*, η_2), we need to consider two different cases.

<u>*Case 1*</u>. $A_{\eta_1} \neq \emptyset$. Denote

$$\tau_2^* \stackrel{\triangle}{=} \inf\{t \geq \tau_1^* : Y_t^{\eta_1} = \max_{i\in A_{\eta_1}} h_{\eta_1,i}(t, Y_t^i)\} \wedge \lambda_2,$$

and, when $\tau_2^* < \lambda_2$, let $\eta_2 \in A_{\eta_1}$ such that $Y_{\tau_2^*}^{\eta_1} = h_{\eta_1,\eta_2}(\tau_2^*, Y_{\tau_2^*}^{\eta_2})$. Then $\overrightarrow{Y}$ satisfies a system with $m_1 - 1$ reflections over $[\tau_1^*, \tau_2^*]$, where the η_1-th equation has no reflection.

<u>*Case 2*</u>. $A_{\eta_1} = \emptyset$. In this case, the η_1-th equation has no reflection. Note that $Y_{\tau_1^*}^{\eta_0} = h_{\eta_0,\eta_1}(\tau_1^*, Y_{\tau_1^*}^{\eta_1})$. As in (4.9), choose τ_2^* "close" to τ_1^*, then for any $t \in [\tau_1^*, \tau_2^*]$, we have $Y_t^{\eta_0} \approx h_{\eta_0,\eta_1}(\tau_1^*, Y_t^{\eta_1})$. On the other hand, by (4.1) and (3.3) one can see that $Y_{\tau_1^*}^j > h_{j,\eta_0}(\tau_1^*, Y_{\tau_1^*}^{\eta_0})$ for any j such that $\eta_0 \in A_j$. Since τ_2^* is close to τ_1^*, let us assume $Y_t^j > h_{j,\eta_0}(\tau_1^*, Y_t^{\eta_0})$ for $t \in [\tau_1^*, \tau_2^*]$. So approximately, over $[\tau_1^*, \tau_2^*]$, $\{Y^j\}_{j\neq\eta_0}$ satisfy

$$\begin{cases} Y_t^j \approx Y_{\tau_2^*}^j + \int_t^{\tau_2^*} f_j(s, h_{1,\eta_1}(\tau_1^*, Y_s^{\eta_1}), Y_s^2, \cdots, Y_s^m, Z_s^j)ds - \int_t^{\tau_2^*} Z_s^j dB_s + K_{\tau_2^*}^j - K_t^j; \\ Y_t^j \geq \max_{k\in A_j - \{\eta_0\}} h_{j,k}(t, Y_t^k); \quad [Y_t^j - \max_{k\in A_j-\{\eta_0\}} h_{j,k}(t, Y_t^k)]dK_t^j = 0. \end{cases} \tag{4.11}$$

This is a system of $m-1$ equations with m_1-1 reflections, where we remove the equation for Y^{η_0} completely. In order to move forward, we need to define η_2 so that $A_{\eta_2}\neq\emptyset$. It turns out that the best way is to set $\eta_2 \stackrel{\triangle}{=} \eta_0$.

Now we can continue the procedure and define a sequence of (τ_n^*, η_n).

4.2 Construction of Y^δ

The arguments in Section 4.1 is only heuristic. We now make everything rigorous. First, let us introduce the following:

Definition 4.3 *$\delta=(\tau_0,\cdots,\tau_n;\eta_0,\cdots,\eta_n)$ is called an admissible strategy if*

(i) $\lambda_1=\tau_0\le\cdots\le\tau_n\le\lambda_2$ is a sequence of stopping times;

(ii) $\eta_0,\cdots,\eta_n$ are random index taking values in $\{1,\cdots,m\}$ such that $\eta_i\in\mathcal{F}_{\tau_i}$;

(iii) $A_{\eta_0}\neq\emptyset$;

(iv) If $A_{\eta_i}\neq\emptyset$, then $\eta_{i+1}\in A_{\eta_i}$;

(v) If $A_{\eta_i}=\emptyset$, then $\eta_{i+1}\stackrel{\triangle}{=}\eta_{i-1}$.

We note that, unlike in Section 2, here δ must be a finite sequence.

Remark 4.4 *By Definition 4.3 (iii), $A_{\eta_i}=\emptyset$ implies that $i\ge 1$. Then the above (v) makes sense. Moreover, by induction we see in this case $A_{\eta_{i+1}}=A_{\eta_{i-1}}\neq\emptyset$.*

We assume Theorem 4.2 holds for $\mu=m_1-1$ and for any $m\ge m_1$. Now assume $\mu=m_1$. For an admissible strategy δ, we construct $(Y^{\delta,j},Z^{\delta,j})$ as follows.

First, for $t\in[\tau_n,\lambda_2]$ and $j=1,\cdots,m$, set

$$Y_t^{\delta,j}\stackrel{\triangle}{=}Y_t^{0,j},\quad Z_t^{\delta,j}\stackrel{\triangle}{=}Z_t^{0,j},\tag{4.12}$$

where $(Y^{0,j},Z^{0,j})$ is the solution to (3.14) constructed in §2. Then in particular we have

$$Y_{\tau_n}^{\delta,j}\ge\max_{i\in A_j}h_{j,i}(\tau_n,Y_{\tau_n}^{\delta,i}),\quad j=1,\cdots,m.\tag{4.13}$$

For $i=n-1,\cdots,0$, assume we have constructed $Y^{\delta,j}_{\tau_{i+1}-}$ for $j=1,\cdots,m$, which we will do later. Note that $Y^{\delta,j}$ may be discontinuous at τ_{i+1}. Corresponding to the *Case 1* and *Case 2* when we defined (τ_2^*,η_2) in Section 4.1, we define $(Y^{\delta,j},Z^{\delta,j})$ over $[\tau_i,\tau_{i+1})$ in two cases.

Case 1. $A_{\eta_i} \neq \emptyset$. Assume our constructed $Y^{\delta,j}_{\tau_{i+1}-}$ satisfies

$$Y^{\delta,j}_{\tau_{i+1}-} \geq \max_{k\in A_j} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}), \quad j \neq \eta_i. \tag{4.14}$$

Recall (4.10). We consider the following RBSDE by removing the constraint of the η_i-th equation:

$$\begin{cases} Y^{\delta,j}_t = Y^{\delta,j}_{\tau_{i+1}-} + \int_t^{\tau_{i+1}} f_j(s, \overrightarrow{Y}^{\delta}_s, Z^{\delta,j}_s)ds - \int_t^{\tau_{i+1}} Z^{\delta,j}_s dB_s + K^{\delta,j}_{\tau_{i+1}} - K^{\delta,j}_t, \ j \neq \eta_i; \\ Y^{\delta,j}_t \geq \max_{k\in A_j} h_{j,k}(t, Y^{\delta,k}_t); \ [Y^{\delta,j}_t - \max_{k\in A_j} h_{j,k}(t, Y^{\delta,k}_t)]dK^{\delta,j}_t = 0, \ j \neq \eta_i; \\ Y^{\delta,\eta_i}_t = Y^{\delta,\eta_i}_{\tau_{i+1}-} + \int_t^{\tau_{i+1}} f_{\eta_i}(s, \overrightarrow{Y}^{\delta}_s, Z^{\delta,\eta_i}_s)ds - \int_t^{\tau_{i+1}} Z^{\delta,\eta_i}_s dB_s. \end{cases} \tag{4.15}$$

It is obvious that the $f_j, h_{j,i}, A_j$ here satisfy Assumptions 3.1 and 4.1, and (4.14) implies that the terminal conditions of (4.15) satisfy (3.15). Since (4.15) has only $m_1 - 1$ reflections, by induction assumption it has the unique solution $(Y^{\delta,j}, Z^{\delta,j}), j = 1, \cdots, m$ over $[\tau_i, \tau_{i+1})$. ■

Case 2. $A_{\eta_i} = \emptyset$. By Remark 4.4 we have $i \geq 1$ and $A_{\eta_{i-1}} \neq \emptyset$. Assume our constructed $Y^{\delta,j}_{\tau_{i+1}-}$ satisfies

$$Y^{\delta,j}_{\tau_{i+1}-} \geq \max_{k\in A_j - \{\eta_{i-1}\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}), \quad j \neq \eta_{i-1}. \tag{4.16}$$

Recall (4.11). We omit the η_{i-1}-th equation and consider the following $m-1$ dimensional RBSDE with at most $m_1 - 1$ reflections: for $j \neq \eta_{i-1}$,

$$\begin{cases} Y^{\delta,j}_t = Y^{\delta,j}_{\tau_{i+1}-} - \int_t^{\tau_{i+1}} Z^{\delta,j}_s dB_s + K^{\delta,j}_{\tau_{i+1}} - K^{\delta,j}_t \\ \quad + \int_t^{\tau_{i+1}} \tilde{f}_j(s, Y^{\delta,1}_s, \cdots, Y^{\delta,\eta_{i-1}-1}_s, Y^{\delta,\eta_{i+1}-1}_s, \cdots, Y^{\delta,m}_s, Z^{\delta,j}_s)ds; \\ Y^{\delta,j}_t \geq \max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^{\delta,k}_t), \quad [Y^{\delta,j}_t - \max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^{\delta,k}_t)]dK^{\delta,j}_t = 0. \end{cases} \tag{4.17}$$

Here:

$$\begin{aligned} &\tilde{f}_j(t, y_1, \cdots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \cdots, y_n, z) \\ &\triangleq f_j(t, y_1, \cdots, y_{\eta_{i-1}-1}, h_{\eta_{i-1},\eta_i}(\tau_i, y_{\eta_i}), y_{\eta_{i-1}+1}, \cdots, y_n, z). \end{aligned} \tag{4.18}$$

One can easily check that $\tilde{f}_j, h_{j,i}, A_j - \{\eta_{i-1}\}$ here satisfy Assumptions 3.1 and 4.1, and (4.16) implies that the terminal conditions of (4.17) satisfy (3.15). Since RBSDE (4.17) has at most $m_1 - 1$ reflections, by induction assumption it has the unique solution $(Y^{\delta,j}, Z^{\delta,j}), j \neq \eta_{i-1}$, over $[\tau_i, \tau_{i+1})$. We emphasize that $Y^{\delta,\eta_{i-1}}$ is not involved in this case. ∎

It remains to construct $Y^{\delta,j}_{\tau_{i+1}-}$ satisfying (4.14) or (4.16). First, set

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{0,j}_{\tau_n}, \text{ if } i+1=n; \qquad Y^{\delta,j}_{\tau_{i+1}-} \triangleq \xi^j_{\lambda_2}, \text{ if } \tau_{i+1}=\lambda_2. \tag{4.19}$$

By (4.12) and (3.15) we know both (4.14) and (4.16) hold. Now assume $i < n-1$ and $\tau_{i+1} < \lambda_2$. Assume we have solved either (4.15) or (4.17) over $[\tau_{i+1}, \tau_{i+2})$. Again we construct $Y^{\delta,j}_{\tau_{i+1}-}$ in two cases.

<u>*Case 2*</u>. $A_{\eta_i} = \emptyset$. In this case we need to construct $Y^{\delta,j}_{\tau_{i+1}-}$ only for $j \neq \eta_{i-1}$ and to check (4.16). By Remark 4.4 we know $i \geq 1, \eta_{i+1} = \eta_{i-1}$, and $A_{\eta_{i+1}} \neq \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}$ were obtained from (4.15) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy:

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k\in A_j} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_{i+1} = \eta_{i-1}. \tag{4.20}$$

Define

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_{i-1}. \tag{4.21}$$

Then (4.16) follows immediately from (4.20). ∎

<u>*Case 1*</u>. $A_{\eta_i} \neq \emptyset$. In this case we need to construct $Y^{\delta,j}_{\tau_{i+1}-}$ for all j and to check (4.14). We do it in two cases.

<u>*Case 1.1*</u>. $A_{\eta_{i+1}} = \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}, j \neq \eta_i$ were obtained from (4.17) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k\in A_j - \{\eta_i\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_i. \tag{4.22}$$

Define

$$Y^{\delta,j}_{\tau_{i+1}-} \triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_i; \quad Y^{\delta,\eta_i}_{\tau_{i+1}-} \triangleq h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}). \tag{4.23}$$

By (4.22), to prove (4.14) it suffices to show that

$$Y^{\delta,j}_{\tau_{i+1}} \geq h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})), \quad \text{if } \eta_i \in A_j. \tag{4.24}$$

Assume $\eta_i \in A_j$. By Definition 4.3 (iv) and Assumption 4.1 (ii), we have $\eta_{i+1} \in [A_j - \{\eta_i\}] \cup \{j\}$, and

$$h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})) < h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}). \tag{4.25}$$

If $\eta_{i+1} \in A_j - \{\eta_i\}$, then (4.24) follows from (4.22) and (4.25). If $\eta_{i+1} = j$, then (4.24) follows from (3.2) and (4.25). So in both cases (4.24) holds, then so does (4.14). ■

Case 1.2. $A_{\eta_{i+1}} \neq \emptyset$. Then $Y^{\delta,j}_{\tau_{i+1}}$ were obtained from (4.15) over $[\tau_{i+1}, \tau_{i+2})$ and thus satisfy:

$$Y^{\delta,j}_{\tau_{i+1}} \geq \max_{k \in A_j} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}), \quad j \neq \eta_{i+1}. \tag{4.26}$$

Define

$$\begin{aligned} Y^{\delta,j}_{\tau_{i+1}-} &\triangleq Y^{\delta,j}_{\tau_{i+1}}, \quad j \neq \eta_i, \eta_{i+1}; \\ Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} &\triangleq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} \vee \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}); \\ Y^{\delta,\eta_i}_{\tau_{i+1}-} &\triangleq h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}). \end{aligned} \tag{4.27}$$

We now check (4.14) for $j \neq \eta_i$. First, for $j = \eta_{i+1}$, by (4.27),

$$Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} \geq \max_{k \in A_{\eta_{i+1}} - \{\eta_i\}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}).$$

Moreover, if $\eta_i \in A_{\eta_{i+1}}$, by (4.1) and (3.2) we have

$$h_{\eta_{i+1},\eta_i}(\tau_{i+1}, Y^{\delta,\eta_i}_{\tau_{i+1}-}) = h_{\eta_{i+1},\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})) < Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}.$$

So (4.14) holds for $j = \eta_{i+1}$.

It remains to check (4.14) for $j \neq \eta_i, \eta_{i+1}$. By (4.26) and the first line in (4.27) we have

$$Y^{\delta,j}_{\tau_{i+1}-} \geq \max_{k \in A_j - \{\eta_i, \eta_{i+1}\}} h_{j,k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}-}). \tag{4.28}$$

If $\eta_{i+1} \in A_j$, recall the definition of $Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}$ in (4.27). First, by (4.26) we have

$$h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}}) \leq Y^{\delta,j}_{\tau_{i+1}} = Y^{\delta,j}_{\tau_{i+1}-}.$$

Second, for any $k \in A_{\eta_{i+1}} - \{\eta_i\}$, similar to (4.24) one can easily prove

$$h_{j,\eta_{i+1}}(\tau_{i+1}, h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}})) \leq Y^{\delta,j}_{\tau_{i+1}} = Y^{\delta,j}_{\tau_{i+1}-}.$$

Thus, by Assumption 3.1 (iii) we have

$$h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}) \leq Y^{\delta,j}_{\tau_{i+1}-}. \tag{4.29}$$

Finally, if $\eta_i \in A_j$, by Definition 4.3 (iv), Assumption 4.1 (ii), and (4.29), we have $\eta_{i+1} \in A_j \bigcup \{j\}$ and

$$h_{j,\eta_i}(\tau_{i+1}, Y^{\delta,\eta_i}_{\tau_{i+1}-}) = h_{j,\eta_i}(\tau_{i+1}, h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})) < h_{j,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}) \leq Y^{\delta,j}_{\tau_{i+1}-}.$$

This, together with (4.28) and (4.29), proves (4.14) for $j \neq \eta_i, \eta_{i+1}$. ∎

Now repeat the arguments backward in time, we see in each $[\tau_i, \tau_{i+1})$, either (4.15) or (4.17) is well defined and is wellposed. Thus we obtain $Y^{\delta,j}$ over the whole interval $[\lambda_1, \lambda_2]$, with the exception of $Y_t^{\delta,\eta_{i-1}}$ for $t \in [\tau_i, \tau_{i+1})$ when $A_{\eta_i} = \phi$. By applying Corollary 3.5 and comparison theorem repeatedly, one can easily show that:

Lemma 4.5 *Assume Assumptions 3.1 and 4.1 hold, and that Theorem 4.2 is true for $\mu = m_1 - 1$. Then for $\mu = m_1$ and for any admissible strategy δ and any j, we have $Y_t^{\delta,j} \leq Y_t^j$ whenever $Y_t^{\delta,j}$ is well defined.* ∎

4.3 Verification Theorem

We now prove the verification theorem.

Theorem 4.6 *Assume Assumptions 3.1 and 4.1 hold, and that Theorem 4.2 is true for $\mu = m_1 - 1$. Then for $\mu = m_1$ and for any solution Y^j of RBSDE (3.14), we have $Y_{\lambda_1}^j = \operatorname*{esssup}_\delta Y_{\lambda_1}^{\delta,j}$ for all j, where the* esssup *is taken over all admissible strategies δ.*

Proof. We prove the theorem in sevaral steps.

Step 1. Fix $\varepsilon > 0$ and let $D_\varepsilon \triangleq \{i\varepsilon : i = 0, 1, \cdots\}$. We construct an approximately optimal admissible strategy as follows. First, set $\tau_0 \triangleq \lambda_1$ and choose η_0 such that $A_{\eta_0} \neq \emptyset$. For $i = 0, 1, \cdots$, we define (τ_{i+1}, η_{i+1}) in two cases.

<u>Case 1</u>. $A_{\eta_i} \neq \emptyset$. Set

$$\tau_{i+1} \triangleq \inf\{t \geq \tau_i : Y_t^{\eta_i} = \max_{k \in A_{\eta_i}} h_{\eta_i,k}(t, Y_t^k)\} \wedge \lambda_2.$$

If $\tau_{i+1} < \lambda_2$, set $\eta_{i+1} \in A_{\eta_i}$ be the smallest index such that

$$Y_{\tau_{i+1}}^{\eta_i} = h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y_{\tau_{i+1}}^{\eta_{i+1}}). \tag{4.30}$$

Otherwise choose arbitrary $\eta_{i+1} \in A_{\eta_i}$.

<u>Case 2</u>. $A_{\eta_i} = \emptyset$. Since $A_{\eta_0} \neq \emptyset$, we have $i \geq 1$. Set $\eta_{i+1} \triangleq \eta_{i-1}$. If $\tau_i = \lambda_2$, define $\tau_{i+1} \triangleq \lambda_2$. Now assume $\tau_i < \lambda_2$. It is more involved to define τ_{i+1} in this case. By the definition of η_i, one can check that in this case we must have $A_{\eta_{i-1}} \neq \emptyset$, and thus by Case 1, $\eta_i \in A_{\eta_{i-1}}$ and $Y_{\tau_i}^{\eta_{i-1}} = h_{\eta_{i-1},\eta_i}(\tau_i, Y_{\tau_i}^{\eta_i})$. Moreover, by Assumptions 3.1 (iii)

and (4.1 (ii), one can easily see $Y^j_{\tau_i} > h_{j,\eta_{i-1}}(\tau_i, Y^{\eta_{i-1}}_{\tau_i})$ for any j such that $\eta_{i-1} \in A_j$. We now define

$$\tau_{i+1} \triangleq \tau^1_{i+1} \wedge \tau^2_{i+1} \wedge \lambda_2;$$

where τ^1_{i+1} is the smallest number in D_ε such that $\tau^1_{i+1} > \tau_i$; and

$$\tau^2_{i+1} \triangleq \inf\{t > \tau_i : \exists j \ s.t.\ \eta_{i-1} \in A_j, Y^j_t = h_{j,\eta_{i-1}}(t, Y^{\eta_{i-1}}_t)\}.$$

Now set $\delta \triangleq \delta^{n,\varepsilon} \triangleq (\tau_0, \cdots, \tau_n; \eta_0, \cdots, \eta_n)$. Recall Definition 4.3. One can easily check that δ is an admissible strategy.

Step 2. We estimate the errors backward in time. Recall Section 4.2 and denote

$$\Delta Y^j_t \triangleq Y^j_t - Y^{\delta,j}_t.$$

First, by (4.19) it is obvious that

$$|Y^j_{\tau_n} - Y^{\delta,j}_{\tau_n-}| = |\Delta Y^j_{\tau_n}|. \tag{4.31}$$

Now assume $i < n-1$.
Case 1. $A_{\eta_i} \neq \phi$. We claim that

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_i}|^2 \le E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|^2\Big\}. \tag{4.32}$$

In fact, in this case (Y^j, Z^j, K^j) satisfies

$$\begin{cases} Y^j_t = Y^j_{\tau_{i+1}} + \int_t^{\tau_{i+1}} f_j(s, \overrightarrow{Y}_s, Z^j_s)ds - \int_t^{\tau_{i+1}} Z^j_s dB_s + K^j_{\tau_{i+1}} - K^j_t, \ j \neq \eta_i; \\ Y^j_t \ge \max_{k\in A_j} h_{j,k}(t, Y^k_t); \quad [Y^j_t - \max_{k\in A_j} h_{j,k}(t, Y^k_t)]dK^k_t = 0, \ j \neq \eta_i; \\ Y^{\eta_i}_t = Y^{\eta_i}_{\tau_{i+1}} + \int_t^{\tau_{i+1}} f_{\eta_i}(s, \overrightarrow{Y}_s, Z^{\eta_i}_s)ds - \int_t^{\tau_{i+1}} Z^{\eta_i}_s dB_s. \end{cases} \tag{4.33}$$

Compare (4.33) and (4.15). They have only $m_1 - 1$ reflections, thus by induction assumption we can apply Theorem 4.2 (ii) and obtain

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_i}|^2 \le E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{1\le j\le m} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}|^2\Big\}.$$

So to prove (4.33) it suffices to show that

$$\max_{1\le j\le m} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| \le \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|. \tag{4.34}$$

If $\tau_{i+1} = \lambda_2$, then by (4.19), we have

$$|Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = |\xi^j_{\lambda_2} - \xi^j_{\lambda_2}| = 0, \quad \forall j.$$

Thus (4.34) holds.

Now assume $\tau_{i+1} < \lambda_2$. Note that $Y^{\delta,j}_{\tau_{i+1}-}$ is defined by either (4.23) or (4.27). In the former case, by (4.2) we have

$$\max_{j\neq\eta_i} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = \max_{j\neq\eta_i} |\Delta Y^j_{\tau_{i+1}}|;$$
$$|Y^{\eta_i}_{\tau_{i+1}} - Y^{\delta,\eta_i}_{\tau_{i+1}-}| = |h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\eta_{i+1}}_{\tau_{i+1}}) - h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}})| \leq |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|.$$

Since $\eta_{i+1} \in A_{\eta_i}$ and thus $\eta_{i+1} \neq \eta_i$. We prove (4.34) in this case.

In the latter case, that is, $Y^{\delta,j}_{\tau_{i+1}-}$ is defined by (4.27). We first have

$$\max_{j\neq\eta_i,\eta_{i+1}} |Y^j_{\tau_{i+1}} - Y^{\delta,j}_{\tau_{i+1}-}| = \max_{j\neq\eta_i,\eta_{i+1}} |\Delta Y^j_{\tau_{i+1}}|.$$

Next, for $j = \eta_{i+1}$, by Lemma 4.5 and Assumption 3.1 (iii) we have

$$Y^{\eta_{i+1}}_{\tau_{i+1}} \geq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} \quad \text{and} \quad Y^{\eta_{i+1}}_{\tau_{i+1}} \geq \max_{k\in A_{\eta_{i+1}}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^k_{\tau_{i+1}}) \geq \max_{k\in A_{\eta_{i+1}}} h_{\eta_{i+1},k}(\tau_{i+1}, Y^{\delta,k}_{\tau_{i+1}}).$$

Then $Y^{\eta_{i+1}}_{\tau_{i+1}} \geq Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}$. Therefore,

$$|Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}| = Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-} \leq Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}} = |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|.$$

Finally, for $j = \eta_i$, by Assumption 4.1 (iii) we have and (4.1), we have

$$\begin{aligned} |Y^{\eta_i}_{\tau_{i+1}} - Y^{\delta,\eta_i}_{\tau_{i+1}-}| &= |h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\eta_{i+1}}_{\tau_{i+1}}) - h_{\eta_i,\eta_{i+1}}(\tau_{i+1}, Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-})| \\ &\leq |Y^{\eta_{i+1}}_{\tau_{i+1}} - Y^{\delta,\eta_{i+1}}_{\tau_{i+1}-}| \leq |\Delta Y^{\eta_{i+1}}_{\tau_{i+1}}|. \end{aligned}$$

Thus (4.34) also holds.

<u>*Case 2*</u>. $A_{\eta_i} = \phi$. In this case $(Y^j, Z^j, K^j), j \neq \eta_{i-1}$ satisfies

$$\begin{cases} Y^j_t = Y^j_{\tau_{i+1}} - \int_t^{\tau_{i+1}} Z^j_s dB_s + K^j_{\tau_{i+1}} - K^j_t \\ \qquad + \int_t^{\tau_{i+1}} \hat{f}_j(s, Y^1_s, \cdots, Y^{\eta_{i-1}-1}_s, Y^{\eta_{i-1}+1}_s, \cdots, Y^m_s, Z^j_s)ds; \\ Y^j_t \geq \max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^k_t); \quad [Y^j_t - \max_{k\in A_j-\{\eta_{i-1}\}} h_{j,k}(t, Y^k_t)]dK^k_t = 0; \end{cases} \tag{4.35}$$

where, recalling (4.18),

$$\begin{aligned}&\hat{f}_j(t, y_1, \cdots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \cdots, y_n, z) \\ &\quad\triangleq \tilde{f}_j(t, y_1, \cdots, y_{\eta_{i-1}-1}, y_{\eta_{i-1}+1}, \cdots, y_n, z) + I_t^j;\end{aligned} \tag{4.36}$$

$$\begin{aligned}I_t^j &\triangleq f_j(t, \overrightarrow{Y}_t, Z_t^j) \\ &\quad - f_j(t, Y_t^1, \cdots, Y_t^{\eta_{i-1}-1}, h_{\eta_{i-1},\eta_i}(\tau_i, Y_t^{\eta_i}), Y_t^{\eta_{i-1}+1}, \cdots, Y_t^n, Z_t^j).\end{aligned} \tag{4.37}$$

We note that here I_t^j is considered as a random coefficient. Compare (4.35) and (4.17). Recalling (4.21), by induction assumption again we get

$$\max_{j\neq\eta_{i-1}} |\Delta Y_{\tau_i}^j|^2 \leq E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_{i-1}} |\Delta Y_{\tau_{i+1}}^j|^2 + C \sum_{j\neq\eta_{i-1}} \int_{\tau_i}^{\tau_{i+1}} |I_t^j| dt\Big\}. \tag{4.38}$$

Note that $Y_{\tau_i}^{\eta_{i-1}} = h_{\eta_{i-1},\eta_i}(\tau_i, Y_{\tau_i}^{\eta_i})$. Then

$$\begin{aligned}|I_t^j| &\leq C\Big|Y_t^{\eta_{i-1}} - h_{\eta_{i-1},\eta_i}(\tau_i, Y_t^{\eta_i})\Big|^2 \\ &\leq C\Big[|Y_t^{\eta_{i-1}} - Y_{\tau_i}^{\eta_{i-1}}|^2 + |h_{\eta_{i-1},\eta_i}(\tau_i, Y_{\tau_i}^{\eta_i}) - h_{\eta_{i-1},\eta_i}(\tau_i, Y_t^{\eta_i})|^2\Big] \\ &\leq C\Big[|Y_t^{\eta_{i-1}} - Y_{\tau_i}^{\eta_{i-1}}|^2 + |Y_{\tau_i}^{\eta_i} - Y_t^{\eta_i}|^2\Big] \leq C\sum_{k=1}^m |Y_t^k - Y_{\tau_i}^k|^2.\end{aligned}$$

Note that in this case $\tau_{i+1} - \tau_i \leq \varepsilon$. Then

$$|I_t^j| \leq C \sum_{k=1}^m \sup_{\lambda_1\leq t_1<t_2\leq\lambda_2 : t_2-t_1\leq\varepsilon} |Y_{t_1}^k - Y_{t_2}^k|^2 \triangleq I_\varepsilon. \tag{4.39}$$

Thus (4.38) implies

$$\max_{j\neq\eta_{i-1}} |\Delta Y_{\tau_i}^j|^2 \leq E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{1\leq j\leq m} |\Delta Y_{\tau_{i+1}}^j|^2 + I_\varepsilon[\tau_{i+1} - \tau_i]\Big\}. \tag{4.40}$$

Step 3. We claim that, for a.s. ω, $\tau_i = \lambda_2$ for i large enough. We prove it by contradiction. Assume ω is in the set that all $Y_\cdot^j(o)$ and $h_{j,i}(\cdot, \omega, y)$ are continuous and $\tau_i(\omega) < \lambda_2$ for all i. Denote $\tau_\infty \triangleq \lim_{i\to\infty} \tau_i$.

First, it is obvious that there can be only finitely many i such that $A_{\eta_i} = \phi$ and $\tau_{i+1} = \tau_{i+1}^1$.

Second, assume there is an infinite sequence i_k such that $A_{\eta_{i_k}} = \phi$ and $\tau_{i_k+1} = \tau_{i_k+1}^2$. Note that in this case $\eta_{i_k} \in A_{\eta_{i_k-1}}$ and there exists $\hat{\eta}_{i_k+1}$ such that $\eta_{i_k-1} \in A_{\hat{\eta}_{i_k+1}}$. Then

$$Y_{\tau_{i_k}}^{\eta_{i_k-1}} = h_{\eta_{i_k-1},\eta_{i_k}}(\tau_{i_k}, Y_{\tau_{i_k}}^{\eta_{i_k}}); \quad Y_{\tau_{i_k+1}}^{\hat{\eta}_{i_k+1}} = h_{\hat{\eta}_{i_k+1},\eta_{i_k-1}}(\tau_{i_k+1}, Y_{\tau_{i_k+1}}^{\eta_{i_k-1}}). \tag{4.41}$$

The vector $(\hat{\eta}_{i_k+1}, \eta_{i_k-1}, \eta_{i_k})$ can take only finitely many values, then there exist (j_1, j_2, j_3) and an infinite subsequence of i_k, without loss of generality we assume it is the whole sequence i_k, such that $j_2 \in A_{j_1}, j_3 \in A_{j_2}$ and

$$\hat{\eta}_{i_k+1} = j_1, \quad \eta_{i_k-1} = j_2, \quad \eta_{i_k} = j_3, \quad \forall k.$$

By (4.41) we get

$$Y^{j_1}_{\tau_{i_k+1}} = h_{j_1,j_2}(\tau_{i_k+1}, Y^{j_2}_{\tau_{i_k+1}}), \quad Y^{j_2}_{\tau_{i_k}} = h_{j_2,j_3}(\tau_{i_k}, Y^{j_3}_{\tau_{i_k}}), \quad \forall k.$$

Send $k \to \infty$, we have

$$Y^{j_1}_{\tau_\infty} = h_{j_1,j_2}(\tau_\infty, Y^{j_2}_{\tau_\infty}), \quad Y^{j_2}_{\tau_\infty} = h_{j_2,j_3}(\tau_\infty, Y^{j_3}_{\tau_\infty}).$$

Then, by Assumption 4.1 (ii), $j_3 \in A_{j_1} \bigcup \{j_1\}$ and

$$Y^{j_1}_{\tau_\infty} = h_{j_1,j_2}(\tau_\infty, h_{j_2,j_3}(\tau_\infty, Y^{j_3}_{\tau_\infty})) < h_{j_1,j_3}(\tau_\infty, Y^{j_3}_{\tau_\infty}).$$

This contradicts with (3.3). Therefore, there are only finitely many i such that $A_{\eta_i} = \phi$.

Finally, by the above results we must have some n_0 such that $A_{\eta_i} \neq \phi$ for all $i \geq n_0$. Then $\eta_{i+1} \in A_{\eta_i}$ and (4.30) holds for all $i \geq n_0$. We say $(\eta_i, \eta_{i+1}, \cdots, \eta_{i+l-1})$ is a *loop* if they are all different and $\eta_{i+l} = \eta_i$. Since each η_i takes only values $1, \cdots, m$, there are in total finitely many possible loops. Thus there exist $(j_1, \cdots, j_l)$ and an infinite sequence i_k such that $(\eta_{i_k}, \cdots, \eta_{i_k+l}, \eta_{i_k+l}) = (j_1, \cdots, j_l, j_1)$. Therefore, by (4.30), we have

$$Y^{j_1}_{\tau_{i_k+1}} = h_{j_1,j_2}(\tau_{i_k+1}, Y^{j_2}_{\tau_{i_k+1}}), \cdots, Y^{j_{l-1}}_{\tau_{i_k+l-1}} = h_{j_{l-1},j_l}(\tau_{i_k+l-1}, Y^{j_l}_{\tau_{i_k+l-1}}),$$

and $Y^{j_l}_{\tau_{i_k+l}} = h_{j_l,j_1}(\tau_{i_k+l}, Y^{j_1}_{\tau_{i_k+l}})$. Send $k \to \infty$, we get

$$Y^{j_1}_{\tau_\infty} = h_{j_1,j_2}(\tau_\infty, Y^{j_2}_{\tau_\infty}), \cdots, Y^{j_{l-1}}_{\tau_\infty} = h_{j_{l-1},j_l}(\tau_\infty, Y^{j_l}_{\tau_\infty}), Y^{j_l}_{\tau_\infty} = h_{j_l,j_1}(\tau_\infty, Y^{j_1}_{\tau_\infty}).$$

This contradicts with Assumption 3.1 (iv). Therefore, we prove the claim.

Step 4. We are now ready to complete the proof. Given $A_{\eta_i} \neq \emptyset$, if $A_{\eta_{i+1}} = \emptyset$, by (4.32) and (4.40) we have

$$\max_{1 \leq j \leq m} |\Delta Y^j_{\tau_i}|^2 \leq E_{\tau_i}\Big\{e^{C(\tau_{i+2}-\tau_i)} \max_{1 \leq j \leq m} |\Delta Y^j_{\tau_{i+2}}|^2 + I_\varepsilon[\tau_{i+2} - \tau_{i+1}]\Big\}. \tag{4.42}$$

By Definition 4.3 (v), we have $A_{\eta_{i+2}} \neq \emptyset$. Therefore, if $A_{\eta_i} \neq \emptyset$, then either $A_{\eta_{i+1}} \neq \emptyset$ and (4.32) holds, or $A_{\eta_{i+2}} \neq \emptyset$ and (4.42) holds. Since $A_{\eta_0} \neq \emptyset$, one gets immediately that

$$\max_{1\le j\le m} |\Delta Y^j_{\tau_0}|^2 \le CE_{\tau_0}\Big\{\max_{1\le j\le m} |\Delta Y^j_{\tau_n}|^2 + I_\varepsilon\Big\} = CE_{\lambda_1}\Big\{\max_{1\le j\le m} |Y^{0,j}_{\tau_n} - Y^j_{\tau_n}|^2 + I_\varepsilon\Big\}.$$

First send $n \to \infty$. Since $\tau_n \to \lambda_2$, we get $Y^{0,j}_{\tau_n} \to \xi^j_{\lambda_2}$ and $Y^j_{\tau_n} \to \xi^j_{\lambda_2}$. By Dominating Convergence Theorem we have

$$\lim_{n\to\infty} \max_{1\le j\le m} |\Delta Y^j_{\lambda_1}|^2 \le CE_{\lambda_1}\{I_\varepsilon\}.$$

Now send $\varepsilon \to 0$. Since Y^j is continuous, by Dominating Convergence Theorem again we get

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \max_{1\le j\le m} |\Delta Y^j_{\lambda_1}|^2 = 0.$$

This, together with Lemma 4.5, proves the theorem. ■

4.4 Proof of Theorem 4.2

As mentioned before, we prove the theorem by induction on μ. When $\mu = 0$, (3.14) is an m-dimensional BSDE without reflections. Then (i) holds, and by standard arguments one can easily prove (ii).

Assume Theorem 4.2 holds for $\mu = m_1 - 1$. Now assume $\mu = m_1$.

(i) By Theorem 4.6, $Y^j_{\lambda_1}$ is unique. Similarly Y^j_t is unique for any $t \in [\lambda_1, \lambda_2]$. By the uniqueness of the Doob-Meyer decomposition we get Z^j is unique, which further implies the uniqueness of K^j immediately.

(ii) For any admissible strategy δ, define $\tilde Y^{\delta,j}$ similarly and denote

$$\Delta Y^{\delta,j}_t \stackrel{\triangle}{=} Y^{\delta,j}_t - \tilde Y^{\delta,j}_t.$$

If $A_{\eta_i} \neq \emptyset$, recalling (4.15), (4.23), and (4.27), by induction we have:

$$\max_{1\le j\le m} |\Delta Y^{\delta,j}_{\tau_i}|^2 \le E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{j\neq\eta_i} |\Delta Y^{\delta,j}_{\tau_{i+1}}|^2 + C\int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt\Big\}.$$

If $A_{\eta_i} = \emptyset$, recalling (4.17) and (4.21), by induction we have:

$$\max_{j\neq\eta_{i-1}} |\Delta Y^{\delta,j}_{\tau_i}|^2 \le E_{\tau_i}\Big\{e^{C(\tau_{i+1}-\tau_i)} \max_{1\le j\le m} |\Delta Y^{\delta,j}_{\tau_{i+1}}|^2 + C\int_{\tau_i}^{\tau_{i+1}} \|\Delta f_t\|^2 dt\Big\}.$$

Note that $A_{\eta_0} \neq \emptyset$. Applying the above estimates repeatedly we get:

$$\max_{1\leq j\leq m} |\Delta Y^{\delta,j}_{\lambda_1}|^2 \leq E_{\lambda_1}\Big\{e^{C(\lambda_2-\lambda_1)} \max_{1\leq j\leq m} |\Delta \xi^j_{\lambda_2}|^2 + C\int_{\lambda_1}^{\lambda_2} \|\Delta f_t\|^2 dt\Big\}, \quad \forall \delta.$$

Then (ii) follows from Theorem 4.6 immediately. ■

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