EE549: Problem Set #1
Due Wednesday, Jan. 17

I. Problem 1 — Input/Output Examples

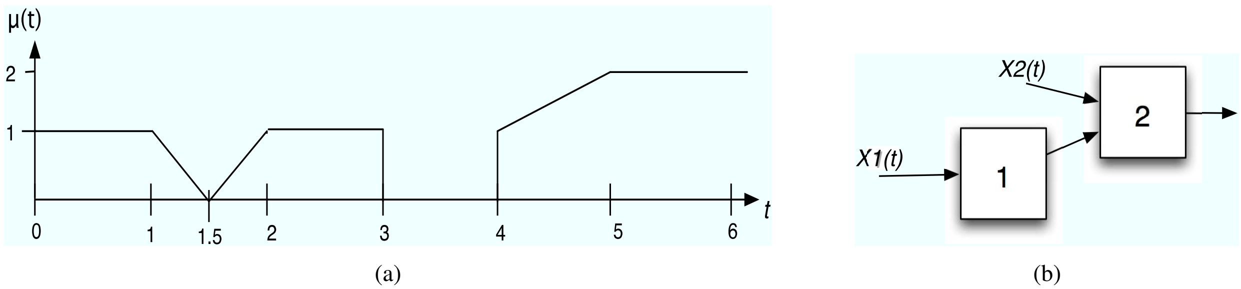


Fig. 1. (a) An example $\mu(t)$ function. (b) A tandem of 2 queues.

a) Three packets of size 1 unit arrive to a queue at times $t = 0.5, 1, 4.6$, and a packet of size 2 units arrives at time $t = 2.7$. The queue is initially empty at time 0, and has a $\mu(t)$ function as shown in Fig. 1(a). Plot the resulting unfinished work function $U(t)$. Assuming FIFO service and that packets do not depart the system until their last bit departs, plot the resulting packet departure function $D(t)$.

b) Consider a tandem of two queues as shown in Fig. 1(b). Both queues have constant server rates $\mu = 1$ kilobyte/second. Packets arrive from stream $X_1(t)$ at times $t = 0.5, 2, 4.5$ seconds. Packets arrive from stream $X_2(t)$ at times $t = 2.5, 4$ seconds. All packets have fixed lengths of 1 kilobyte. Plot the departure functions $D_1(t)$ and $D_2(t)$.

c) Remove the first queue and assume the $X_1(t)$ stream enters into the final queue after a delay of 1 second (so that the total input to the final queue is the sum process $X_1(t-1) + X_2(t)$). Plot the departure function and compare to part b. Explain. Is packet ordering preserved?

II. Problem 2 — Correlations In The Output Process

Consider a queue that is empty at time zero. Packets arrive according to a Poisson process of rate λ (so that inter-arrival times are i.i.d. and exponentially distributed with mean $1/\lambda$). Suppose that all service times are equal to a constant T. Suppose that the first packet arrives at time x (so that it departs at time $x + T$), and let τ_2 and τ_3 represent the interarrival times for packets 2 and 3. Let υ_2 and υ_3 represent the *inter-departure* times for packets 2 and 3 (so that υ_2 is the difference between the departure times of packets 1 and 2, etc.). Show that, despite the fact that interarrival times τ_2 and τ_3 are independent, the inter-departure times υ_2, υ_3 are not independent. (Hint: Compute $Pr[\upsilon_3 > T | \upsilon_2 = T]$, and $Pr[\upsilon_3 > T | \upsilon_2 > T]$).

III. Problem 3 — True/False

For all problems, indicate whether the result is TRUE or FALSE. If TRUE, explain why. If FALSE, provide a counter-example.

1) For a single server, work conserving queue with server process $\mu(t)$, we have for any two times t_1, t_2 such that $t_1 < t_2$

$$U(t_2) = U(t_1) + X(t_1, t_2] - \int_{t_1}^{t_2} \mu(\tau) d\tau$$

2) Suppose $X_1(t)$ and $X_2(t)$ are two arrival processes such that $X_1(t) \leq X_2(t)$ for all t. $X_1(t)$ enters a work conserving queue that is initially empty and has a constant server rate μ. $X_2(t)$ also enters a work conserving queue that is initially empty and has a constant server rate μ. Then $U_1(t) \leq U_2(t)$ for all t.

3) For any times $t_1 < t_2$, we have:

$$X[t_1, t_2] = X(t_2) - X(t_1) + (X(t_1^+) - X(t_1^-))$$

4) Let $X(t)$ be an arrival process, and let $\{a_1, a_2, \ldots, a_n, \ldots\}$ represent the sequence of packet arrival times and $\{B_1, B_2, \ldots, B_n, \ldots\}$ represent the sequence of packet lengths. Then for every n, we have:

$$B_n \leq X(a_n + 5) - X(a_n - 5)$$

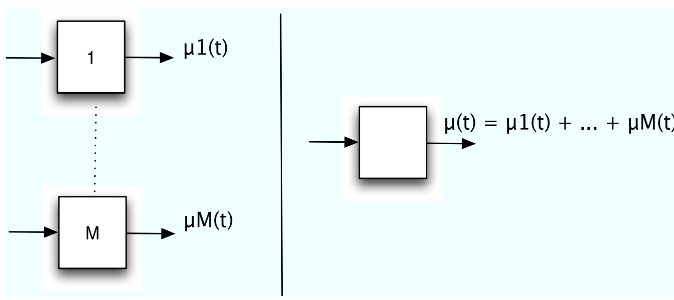
IV. Problem 4 — Inequality Comparison

Consider a queue with server process $\mu(t)$ and input process $X(t)$, and let $U_1(t)$ represent the resulting unfinished work function. Now let $U_2(t)$ represent the unfinished work in another queue with the same server rate $\mu(t)$ but with an input stream $X(t) + Z(t)$, where $Z(t)$ is any other arrival process. Both queues are initially empty, and both are work conserving. We want to show that $U_1(t) \leq U_2(t)$ for all $t \geq 0$.

a) Fix a time $t \geq 0$. Show that the result holds in the special case when $U_1(t) = 0$.

b) Now show that the result also holds in the opposite case when $U_1(t) > 0$.

V. Problem 5 — Shortest Packet First and the Multiplexing Inequality



Consider a parallel set of M queues with time varying processing rates $\mu_1(t), \ldots, \mu_M(t)$, and compare with a single queue of rate $\mu(t) = \mu_1(t) + \ldots + \mu_M(t)$. Both systems are initially empty when a set of K packets simultaneously arrive at time 0. No more packets ever arrive. The lengths of each packet are written in order of size: $B_1 \leq B_2 \leq \ldots \leq B_K$.

- In the multi-server system, the packets are routed amongst the M servers according to some arbitrary routing policy.
- In the single-server system, the packets are served in Shortest-Packet-First (SPF) order (so that B_1 is served first, then B_2, etc.).

We want to show that $L_{multi}(t) \geq L_{single}(t)$ for all t, where $L_{multi}(t)$ and $L_{single}(t)$ represent the number of packets in the multi-server system and the single-server system, respectively, at time t.

a) Show that $L_{multi}(t) \geq L_{single}(t)$ for all t in the special case $K = 1$ (i.e., only 1 packet).

b) Show that, for any K, the single queue system empties no later than the multi-queue system.

c) Argue that, for any $n \in \{1, \ldots, K-1\}$, n packets have departed the single queue system if and only if:

$$\sum_{i=1}^{n} B_i \leq Y_{single}(t) < \sum_{i=1}^{n+1} B_i$$

d) Prove that $L_{multi}(t) \geq L_{single}(t)$ for all t (for arbitrary K).