Quiz 1 — EE 599
Stochastic Network Optimization
Wednesday, Feb. 21, 2007

INSTRUCTIONS

Put your name on the top right corner of this booklet. This quiz is closed book and closed notes. No Calculators or laptops allowed. There are four questions, with point values given below:

1) Lyapunov Drift (25 points)
2) Approximate Scheduling (20 points)
3) Necessary Conditions for Stability (20 points)
4) Computing and Minimizing the Drift (35 points)

FORMULAS

1) Discrete Time Queues:
   - $U(t+1) = \max[U(t) - \mu(t), 0] + A(t)$
   - $U(t) = U(0) + \sum_{\tau=0}^{t-1} A(\tau) - \sum_{\tau=0}^{t-1} \tilde{\mu}(\tau)$
2) Strong Stability:
   - A queue $U(t)$ is defined to be *strongly stable* if: $\limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1} \mathbb{E}\left\{U(\tau)\right\} < \infty$.
   - If a queue $U(t)$ is strongly stable and $\mu(t) \leq \mu_{max}$ for all t (for some finite constant μ_{max}), then:
   $$\lim_{t\to\infty} \frac{\mathbb{E}\left\{U(t)\right\}}{t} = 0$$
3) Lyapunov Drift:
   - $L(\boldsymbol{U}(t))$ is a non-negative function of queue backlog vector $\boldsymbol{U}(t)$. The conditional Lyapunov drift is defined: $\Delta(\boldsymbol{U}(t)) \triangleq \mathbb{E}\left\{L(\boldsymbol{U}(t+1)) - L(\boldsymbol{U}(t)) \mid \boldsymbol{U}(t)\right\}$.
   - If for all t and all possible $\boldsymbol{U}(t)$ we have:
   $$\Delta(\boldsymbol{U}(t)) \leq B - \epsilon \sum_{k=1}^{K} U_k(t)$$
   for some $\epsilon > 0$ and $B > 0$, then:
   $$\overline{\sum_{k=1}^{K} U_k} \leq \frac{B}{\epsilon}$$
   where:
   $$\overline{\sum_{k=1}^{K} U_k} \triangleq \limsup_{t\to\infty} \frac{1}{t}\sum_{\tau=0}^{t-1}\sum_{k=1}^{K} \mathbb{E}\left\{U_k(\tau)\right\}$$
4) For any values $U \geq 0, \mu \geq 0, A \geq 0$, we have:
$$(\max[U-\mu, 0] + A)^2 \leq U^2 + \mu^2 + A^2 - 2U(\mu - A)$$

I. LYAPUNOV DRIFT (25 POINTS)

Let $\boldsymbol{U}(t) = (U_1(t), U_2(t), \ldots, U_K(t))$ be a vector process that evolves in discrete time according to some probability law (for timeslots $t \in \{0, 1, 2, \ldots\}$). Let $L(\boldsymbol{U})$ be a non-negative function of the $\boldsymbol{U}$ vector, and define the conditional Lyapunov drift $\Delta(\boldsymbol{U}(t))$ as follows:

$$\Delta(\boldsymbol{U}(t)) \triangleq \mathbb{E}\left\{L(\boldsymbol{U}(t+1)) - L(\boldsymbol{U}(t)) \mid \boldsymbol{U}(t)\right\}$$

Suppose there are processes $f(t)$ and $g(t)$ (that may depend on $\boldsymbol{U}(t)$) such that:

$$\Delta(\boldsymbol{U}(t)) \leq \mathbb{E}\left\{g(t) - f(t) \mid \boldsymbol{U}(t)\right\} \quad \text{for all } t \text{ and all possible } \boldsymbol{U}(t) \tag{1}$$

a) Suppose that the drift condition (1) holds. Prove that:

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E}\left\{f(\tau)\right\} \leq \limsup_{t\to\infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E}\left\{g(\tau)\right\}$$

Be sure to write helpful phrases/sentences that explain your steps.

b) Suppose that the drift condition (1) holds for the particular processes:

$$\begin{aligned} g(t) &= B \text{ for all } t \\ f(t) &= \epsilon \sum_{k=1}^{K} U_k(t) \text{ for all } t \end{aligned}$$

for some constants $B > 0$ and $\epsilon > 0$. What can be concluded (from part (a)) for this particular case?

II. APPROXIMATE SCHEDULING (20 POINTS)

Let $\boldsymbol{U}(t) = (U_1(t), \ldots, U_K(t))$ be a vector process of queue backlogs that evolves in discrete time. The stochastics of $\boldsymbol{U}(t)$ depend upon the *control decision* $I(t)$ that is made every slot, where $I(t)$ is subject to the constraint $I(t) \in \mathcal{I}$ for all t. Suppose there is a non-negative function $L(\boldsymbol{U}(t))$ with a conditional drift that satisfies for all slots t and all possible $\boldsymbol{U}(t)$:

$$\Delta(\boldsymbol{U}(t)) \leq B - \mathbb{E}\left\{\Phi(I(t), \boldsymbol{U}(t)) \,|\, \boldsymbol{U}(t)\right\}$$

for some constant $B > 0$ and some function $\Phi(I(t), \boldsymbol{U}(t))$.

Suppose there are three different control policies, namely $I^*(t)$, $I^{opt}(t)$, and $\hat{I}(t)$, that each satisfy the constraint $I(t) \in \mathcal{I}$ for all t, and that satisfy for all t and all possible $\boldsymbol{U}(t)$:

- $I^*(t)$ satisfies (for some constant $\epsilon > 0$):

$$\mathbb{E}\left\{\Phi(I^*(t), \boldsymbol{U}(t)) \,|\, \boldsymbol{U}(t)\right\} = \epsilon \sum_{k=1}^{K} U_k(t) \tag{2}$$

- $I^{opt}(t)$ observes $\boldsymbol{U}(t)$ and is chosen to maximize $\Phi(I(t), \boldsymbol{U}(t))$ subject to $I(t) \in \mathcal{I}$.
- $\hat{I}(t)$ is a policy that approximates $I^{opt}(t)$, and satisfies (for some constant $C > 0$ and for all $\boldsymbol{U}(t)$):

$$\Phi(I^{opt}(t), \boldsymbol{U}(t)) \geq \Phi(\hat{I}(t), \boldsymbol{U}(t)) \geq \Phi(I^{opt}(t), \boldsymbol{U}(t)) - C \tag{3}$$

Suppose we implement policy $\hat{I}(t)$ for all time t, yielding conditional Lyapunov drift $\hat{\Delta}(\boldsymbol{U}(t))$. Compute a bound on $\overline{\sum_k U_k}$, the time average expected total queue backlog under this policy.

The 2-queue system described on this page is used in the next two problems (problems 3 and 4).

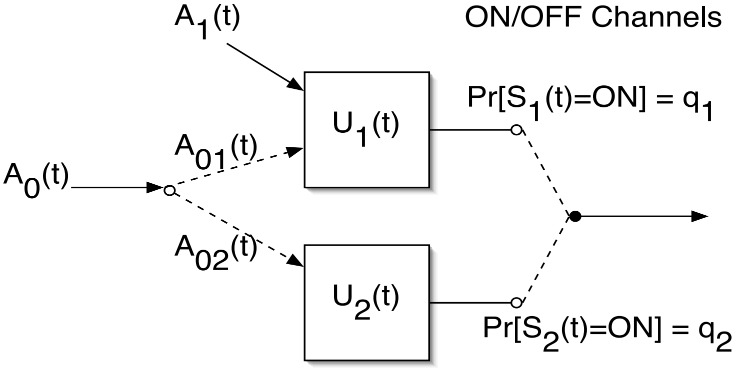

Fig. 1. A two queue system with a single server and ON/OFF channels.

Let $U_1(t)$ and $U_2(t)$ represent the (integer) number of packets in queues 1 and 2 of the two-queue system shown in Fig. 1. The system operates in discrete time with timeslots $t \in \{0, 1, 2, \ldots\}$. The arrival processes $A_0(t)$ and $A_1(t)$ represent the number of packets arriving from stream 0 and 1, respectively. These arrivals are Bernoulli, being independent over slots with $A_0(t) \in \{0, 1\}$ and $A_1(t) \in \{0, 1\}$ for all t. Packets from $A_0(t)$ must be routed immediately to either queue 1 or queue 2. Packets from $A_1(t)$ immediately enter queue 1. The channel processes $S_1(t)$ and $S_2(t)$ represent the ON/OFF states of channels 1 and 2 during slot t, and are i.i.d. over slots. There is a single server that must be allocated to either channel 1 or channel 2.

Every timeslot, the network controller observes the channel states $S_1(t), S_2(t)$ and the new arrivals $A_0(t)$, and makes routing decisions $\{A_{01}(t), A_{02}(t)\}$ and server decisions $\{x_1(t), x_2(t)\}$, where:

1) $A_{01}(t)$ and $A_{02}(t)$ are the number of stream 0 packets routed to queues 1 and 2, respectively, during slot t, with constraints:

$$A_{01}(t) \in \{0, 1\} \ , \ A_{02}(t) \in \{0, 1\}$$
$$A_{01}(t) + A_{02}(t) = A_0(t)$$

2) $x_1(t)$ and $x_2(t)$ are the decisions about which queue to serve (where $x_i(t) = 1$ if we serve queue i during slot t, and $x_i(t) = 0$ otherwise), with constraints:

$$x_1(t) \in \{0, 1\} \ , \ x_2(t) \in \{0, 1\}$$
$$x_1(t) + x_2(t) \leq 1$$

Each queue can serve exactly one packet per slot if a server is allocated to it and if its corresponding channel is ON, else it serves 0 packets. We thus have transmission rates $\mu_1(t)$, $\mu_2(t)$, where:

$$\mu_i(t) = C_i(x_i(t), S_i(t)) \triangleq \begin{cases} 1 & \text{if } S_i(t) = ON \text{ and } x_i(t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

III. NECESSARY CONDITION FOR STABILITY (20 POINTS)

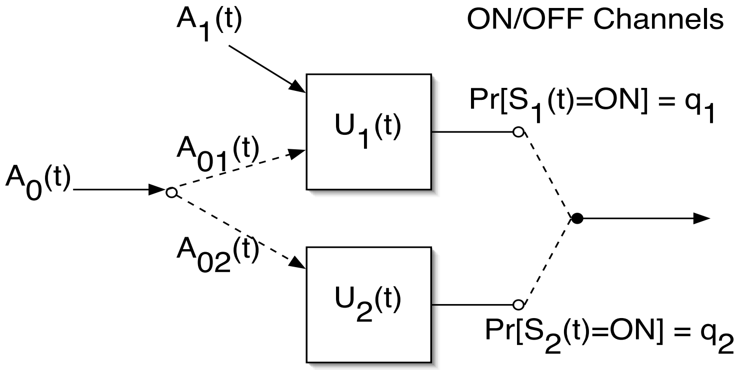

Fig. 2. A two queue system with a single server and ON/OFF channels.

For the 2-queue system (shown again in Fig. 2 above), assume that arrivals $A_0(t)$ and $A_1(t)$ are independent Bernoulli processes that are i.i.d. over slots, and that channel states $S_1(t)$ and $S_2(t)$ are independent ON/OFF processes that are i.i.d. over slots, with:

- $\mathbb{E}\{A_0(t)\} = \lambda_0$, $\mathbb{E}\{A_1(t)\} = \lambda_1$
- $Pr[S_1(t) = ON] = q_1$, $Pr[S_2(t) = ON] = q_2$

Prove that the following constraint is *necessary* for the existence of a joint routing and server allocation algorithm that makes both queues strongly stable:

$$\lambda_0 + \lambda_1 \leq q_2 + (1 - q_2)q_1$$

Proof of Necessity (make sure to write phrases/sentences that justify your steps):

IV. COMPUTING AND MINIMIZING THE DRIFT (35 POINTS)

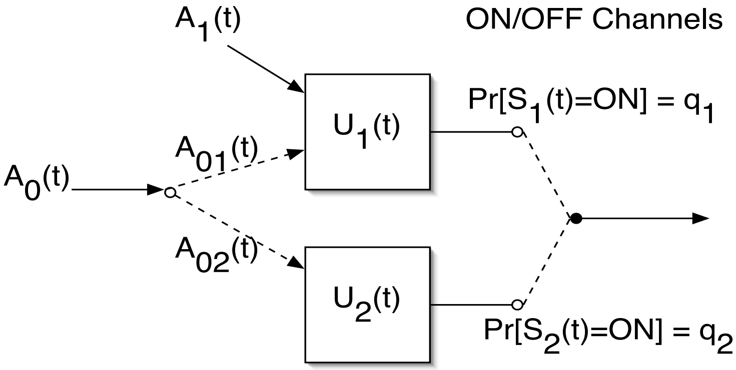

Fig. 3. A two queue system with a single server and ON/OFF channels.

Consider the same system with i.i.d. arrivals and channels (shown again in Fig. 3), and recall that $A_0(t)$ and $A_1(t)$ are i.i.d. Bernoulli with $A_i(t) \in \{0,1\}$ and $\mathbb{E}\{A_0(t)\} = \lambda_0$, $\mathbb{E}\{A_1(t)\} = \lambda_1$ for all t.

a) Write a queueing equation that expresses $U_1(t+1)$ and $U_2(t+1)$ in terms of:

$$U_1(t), U_2(t), \mu_1(t), \mu_2(t), A_{01}(t), A_{02}(t), A_1(t)$$

b) Let $\boldsymbol{U}(t) = (U_1(t), U_2(t))$, and define $L(\boldsymbol{U}(t)) \triangleq U_1(t)^2 + U_2(t)^2$. Define:

$$\Delta(\boldsymbol{U}(t)) \triangleq \mathbb{E}\left\{L(\boldsymbol{U}(t+1)) - L(\boldsymbol{U}(t)) \mid \boldsymbol{U}(t)\right\}$$

Compute $\Delta(\boldsymbol{U}(t))$ for a general policy, and show it has the form:

$$\Delta(\boldsymbol{U}(t)) \leq B - 2\mathbb{E}\left\{\sum_{i=1}^{2} U_i(t)\left[\mu_i(t) - f_i(A_{01}(t), A_{02}(t))\right] \mid \boldsymbol{U}(t)\right\}$$

You must show your work, and explicitly compute the constant B and the functions $f_1(\cdot)$ and $f_2(\cdot)$ in terms of λ_1, λ_2, q_1, q_2 (there is more room on the next page).

c) Recall that routing decisions $A_{01}(t)$ and $A_{02}(t)$ and server decisions $x_1(t)$ and $x_2(t)$ must satisfy:

$$A_{01}(t) \in \{0,1\} \ , \ A_{02}(t) \in \{0,1\} \ , \ A_{01}(t) + A_{02}(t) = A_0(t)$$
$$x_1(t) \in \{0,1\} \ , \ x_2(t) \in \{0,1\} \ , \ x_1(t) + x_2(t) \leq 1$$

Recall that the transmission rates are given by (for $i \in \{1,2\}$):

$$\mu_i(t) = C_i(x_i(t), S_i(t)) \triangleq \begin{cases} 1 & \text{if } S_i(t) = ON \text{ and } x_i(t) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Design a control policy that chooses server decisions $x_1(t), x_2(t)$ and routing decisions $A_{01}(t)$ and $A_{02}(t)$ that minimizes the right hand side of your drift expression in part (b) over all possible control policies that satisfy the constraints. You should state the policy in its simplest form, observing that routing and server decisions can be *decoupled*.