

Summary of Results for First 3 Lectures

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We have followed Section 3.1 and 4.4 of the text [1].

I. DISCRETE TIME QUEUES AND STABILITY (SECTION 3.1 OF TEXT)

A. Discrete Time Queues

One-Step Dynamic Equation for Discrete Time Queues:

$$U(t+1) = \max[U(t) - \mu(t), 0] + A(t) \quad (1)$$

Sample path inequality:

$$U(t) \geq \sum_{\tau=0}^{t-1} A(\tau) - \sum_{\tau=0}^{t-1} \mu(\tau) \quad \text{for all } t \quad (2)$$

B. Strong Stability

Definition 1: A discrete time queue with backlog process $U(t)$ is *strongly stable* if:

$$\lim_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E} \{U(\tau)\} < \infty$$

Definition 2: A network of discrete time queues is strongly stable if all individual queues are strongly stable.

Lemma 1: Suppose that $\mu(t) \leq \mu_{max}$ for all t (for some finite bound μ_{max}). Then if $U(t)$ is strongly stable, we have:

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E} \{U(t)\}}{t} = 0 \quad (3)$$

Proof: The proof is beyond the scope of this course. The interested reader can find the proof in the appendix of [2]. \square

We had definitions of $\mu(t)$ and $A(t)$ being *admissible* with rates $\bar{\mu}$ and λ , respectively (given in Section 3.1 of text). The $\mu(t) \leq \mu_{max}$ assumption is part of admissibility, and will be assumed to hold throughout this course. Note that if $\{\mu(t)\}_{t=0}^{\infty}$ is an i.i.d. sequence with a bounded μ_{max} , then it is admissible. Likewise, if $\{A(t)\}_{t=0}^{\infty}$ is an i.i.d. sequence with bounded first and second moments, then it is admissible.

Theorem 1: (Stability Theorem) If $A(t)$ is admissible with rate λ and $\mu(t)$ is admissible with rate $\bar{\mu}$, then:

- (a) Strong Stability implies that $\lambda \leq \bar{\mu}$ (and so $\lambda \leq \bar{\mu}$ is *necessary* for strong stability).
- (b) $\lambda < \bar{\mu}$ implies strong stability (and so $\lambda < \bar{\mu}$ is *sufficient* for strong stability).

Note there is a “singularity” between necessity and sufficiency for $\lambda = \bar{\mu}$. In this case, there are examples where the queue is strongly stable, and there are other examples where the queue is not strongly stable.

Proof: The proof of part (a) uses (2) together with the fact that (3) holds for strongly stable queues. You are expected to know the proof.

The proof of (b) was done in class for the i.i.d. case, using Lyapunov drift theory. You are also expected to know the proof for this i.i.d. case. The proof for the non-i.i.d. case is given in Section 4.4 of the text using the idea of T -slot Lyapunov drift. \square

Note that throughout this course we shall use the term “stability” to refer to strong stability.

II. LYAPUNOV DRIFT (SECTION 4.4 OF TEXT)

Let $\mathbf{U}(t) = (U_1(t), U_2(t), \dots, U_K(t))$ represent a vector process of queue backlogs that evolve in discrete time $t \in \{0, 1, 2, \dots\}$. Let $L(\mathbf{U})$ represent a non-negative function, called a *Lyapunov function*, of the queue backlog vector. Define the one-step conditional Lyapunov drift as follows:

$$\Delta(\mathbf{U}(t)) \triangleq \mathbb{E} \{L(\mathbf{U}(t+1)) - L(\mathbf{U}(t)) \mid \mathbf{U}(t)\} \quad (4)$$

Theorem 2: (Lyapunov Drift) Suppose that $\mathbf{U}(t)$ evolves according to some probability law, and suppose there exists a non-negative function $L(\mathbf{U})$ and constants $B < \infty$ and $\epsilon > 0$ such that for all timeslots t and all possible values of $\mathbf{U}(t)$, we have:

$$\Delta(\mathbf{U}(t)) \leq B - \epsilon \sum_{k=1}^K U_k(t) \quad (5)$$

Then the queueing network is strongly stable (i.e., all queues are strongly stable), and:

$$\overline{\sum_{k=1}^K U_k} \triangleq \limsup_{t \rightarrow \infty} \frac{1}{t} \sum_{\tau=0}^{t-1} \mathbb{E} \left\{ \sum_{k=1}^K U(\tau) \right\} \leq \frac{B}{\epsilon}$$

Proof: You are expected to know the proof. The proof uses 2 main concepts:

- 1) Iterated Expectations: $\mathbb{E}\{X\} = \mathbb{E}\{\mathbb{E}\{X|Y\}\}$
- 2) Telescoping Sums: $\sum_{m=0}^{M-1} [f(m+1) - f(m)] = f(M) - f(0)$

□

A typical Lyapunov function that is very useful is the following quadratic function:

$$L(\mathbf{U}) = \sum_{k=1}^K U_k^2$$

A fact that is often useful in dealing with quadratic Lyapunov functions: If $U \geq 0, \mu \geq 0, A \geq 0$ then:

$$(\max[U - \mu, 0] + A)^2 \leq U^2 + \mu^2 + A^2 - 2U(\mu - A)$$

This fact is used together with the Lyapunov Drift Theorem to prove part (b) of Theorem 1.

III. SOME COMMENTS ABOUT LYAPUNOV FUNCTIONS, DELAY, AND COMPLEXITY

Lyapunov drift for network stability is first used in [3] for multi-hop networks, and in [4] for opportunistic downlink scheduling. Related quadratic Lyapunov functions are used to make stability and delay claims for $N \times N$ packet switches in [5] and for multi-hop mobile networks in [6]. Non-quadratic Lyapunov functions can sometimes be used to make modified or improved statements about delay [7] [8] [9]. Alternative Lyapunov functions via queue groupings can often lead to improved complexity and/or delay bounds, see [10] [11] [12] [13].

Performance optimal Lyapunov networking will be a large part of this course and will likely be useful for your projects. However, we will not get to this for another few weeks. Students can always read ahead in the text, and are also referred to [2] for a writeup that emphasizes average power minimization and virtual power queues, which is not covered in as much detail in the text. See also papers here: <http://www-rcf.usc.edu/~mjneely/stochastic/>

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