Hausman Test and Weak Instruments: Online Appendix

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A Practical Issues of Implementation: $\mathcal{H}_4$

We will assume that all the variables are in fact residuals when regressed on 'included exogenous variables'. To be more specific, let X denote the included exogenous variable matrix. We then obtain:

- Regress the dependent variable on X, and get the residual. Call it y_1.
- Regress the endogenous regressor on X, and get the residual. Call it Y_2.
- Regress the weak IV on X, and get the residual. Call it W.
- Regress the strong IV on X, and get the residual. Call it S.

A.1 General Case

Our algorithm below is based on the following idea. Let $\widehat{Y}_2 = P_Z Y_2$, and note that $W'\left(Y_2 - \widehat{Y}_2\right) = 0$ by construction. Therefore, we can rewrite

$$\mathcal{H}_4 = \widetilde{\sigma}_{\varepsilon,z}^{-2}\left(y_1 - Y_2\widehat{\beta}_z\right)' W \left[W'\left(I - \widehat{Y}_2\left(\widehat{Y}_2'\widehat{Y}_2\right)^{-1}\widehat{Y}_2'\right)W\right]^{-1} W'\left(y_1 - Y_2\widehat{\beta}_z\right).$$

- Computation of $\widehat{\Psi}$:
  1. Regress Y_2 on $Z = [W, S]$, and get the predicted value. Call it $\widehat{Y}_2$.
  2. Regress W on $\widehat{Y}_2$, and get the residual. Call it $\widetilde{W}$.
  3. Let $\widehat{\Psi} = \widetilde{W}'\widetilde{W}$.
- Computation of $y_1 - Y_2\widehat{\beta}_z$:
  1. Obtain the 2SLS $\widehat{\beta}_z$ by using the instrument $Z = [W, S]$.
  2. Obtain the IV residual $\widehat{\varepsilon}_z = y_1 - Y_2\widehat{\beta}_z$.
- Computation of $\widetilde{\sigma}_{\varepsilon,z}^2$:
  1. Using the IV estimator $\widehat{\beta}_z$, get the IV residual $\widehat{\varepsilon}_z = y_1 - Y_2\widehat{\beta}_z$.
  2. Regress the IV residual $\widehat{\varepsilon}_z$ on $Z = [S, W]$, and get the residual $\widetilde{\varepsilon}_z = M_Z\widehat{\varepsilon}_z$.
  3. Calculate $\widetilde{\sigma}_{\varepsilon,z}^2 = \frac{1}{n}\widetilde{\varepsilon}_z'\widetilde{\varepsilon}_z$.
- $\mathcal{H}_4$ can now be calculated as

$$\mathcal{H}_4 = \frac{\widehat{\varepsilon}_z' W \widehat{\Psi}^{-1} W' \widehat{\varepsilon}_z}{\widetilde{\sigma}_{\varepsilon,z}^2}.$$

B Omitted Proof of Section 5

B.1 Proof of Lemma 1

Here denote $\widetilde{Z} = \left[W, \widetilde{S}\right]$. Since $S = W\Gamma_w + \widetilde{S}$,

$$Z = [W, S] = \left[W, \widetilde{S}\right] \begin{bmatrix} I & \Gamma_w \\ 0 & I \end{bmatrix} = \widetilde{Z} \begin{bmatrix} I & \Gamma_w \\ 0 & I \end{bmatrix}.$$

We first write

$$\begin{aligned}
\frac{1}{\sqrt{n}} W' \left(y_1 - Y_2 \widehat{\beta}_z\right) &= \frac{1}{\sqrt{n}} W' \left(\varepsilon - Y_2 \left(\widehat{\beta}_z - \beta\right)\right) \\
&= \frac{1}{\sqrt{n}} W'\varepsilon - \frac{1}{n} W' Y_2 \cdot \sqrt{n} \left(\widehat{\beta}_z - \beta\right) \\
&= \frac{1}{\sqrt{n}} W'\varepsilon - \frac{1}{n} W' Y_2 \cdot \left(\frac{1}{n} Y_2' Z \cdot \left(\frac{1}{n} Z'Z\right)^{-1} \cdot \frac{1}{n} Z' Y_2\right)^{-1} \frac{1}{n} Y_2' Z \cdot \left(\frac{1}{n} Z'Z\right)^{-1} \cdot \frac{1}{\sqrt{n}} Z'\varepsilon \\
&= \frac{1}{\sqrt{n}} W'\varepsilon - \frac{1}{n} W' Y_2 \cdot \left(\frac{1}{n} Y_2' \widetilde{Z} \cdot \left(\frac{1}{n} \widetilde{Z}'\widetilde{Z}\right)^{-1} \cdot \frac{1}{n} \widetilde{Z}' Y_2\right)^{-1} \frac{1}{n} Y_2' \widetilde{Z} \cdot \left(\frac{1}{n} \widetilde{Z}'\widetilde{Z}\right)^{-1} \cdot \frac{1}{\sqrt{n}} \tilde{Z}'\varepsilon \\
&\Rightarrow \mathcal{Z}_{w\varepsilon} - \Sigma_{ww}\Pi_w \cdot \left(\begin{bmatrix} \Pi_w' & \Pi_s' \end{bmatrix} \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{\widetilde{s}\widetilde{s}} \end{bmatrix} \begin{bmatrix} \Pi_w \\ \Pi_s \end{bmatrix}\right)^{-1} \begin{bmatrix} \Pi_w' & \Pi_s' \end{bmatrix} \begin{bmatrix} \mathcal{Z}_{w\varepsilon} \\ \mathcal{Z}_{\widetilde{s}\varepsilon} \end{bmatrix} \\
&= \mathcal{Z}_{w\varepsilon} - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \left(\Pi_w'\mathcal{Z}_{w\varepsilon} + \Pi_s'\mathcal{Z}_{\widetilde{s}\varepsilon}\right) \\
&= \left(I - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_w'\right) \mathcal{Z}_{w\varepsilon} \\
&+ \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_s'\mathcal{Z}_{\widetilde{s}\varepsilon}.
\end{aligned}$$

Further, note that

$$\begin{bmatrix} \mathcal{Z}_{w\varepsilon} \\ \mathcal{Z}_{\widetilde{s}\varepsilon} \end{bmatrix} \sim N\left(0, \sigma_\varepsilon^2 \begin{bmatrix} \Sigma_{ww} & 0 \\ 0 & \Sigma_{\widetilde{s}\widetilde{s}} \end{bmatrix}\right),$$

and

$$\left(\frac{1}{n} Y_2' W\right) \left(\frac{1}{n} W'W\right)^{-1} \to_p \Pi_w'.$$

It follows that

$$\begin{aligned}
\frac{1}{\sqrt{n}} W' \left(y_1 - Y_2\widehat{\beta}_z\right) \Rightarrow &\left(I - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_w'\right) \mathcal{Z}_{w\varepsilon} \\
&+ \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_s'\mathcal{Z}_{\widetilde{s}\varepsilon}
\end{aligned}$$

is asymptotically normal with asymptotic variance equal to σ_ε^2 times

$$\begin{aligned}
\Psi = &\left(I - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_w'\right) \Sigma_{ww} \left(I - \Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_w'\Sigma_{ww}\right) \\
&+ \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1} \Pi_w'\Sigma_{ww}
\end{aligned}$$

or

$$
\begin{aligned}
\Psi =& \left(I - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\right)\Sigma_{ww}\left(I - \Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww}\right) \\
&+ \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\left(\left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right) - \Pi_w'\Sigma_{ww}\Pi_w\right) \\
&\times \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww} \\
=& \left(I - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\right)\Sigma_{ww}\left(I - \Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww}\right) \\
&+ \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww} \\
&- \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww}\Pi_w\left(I - \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww}\right) \\
=& \Sigma_{ww} - \Sigma_{ww}\Pi_w \left(\Pi_w'\Sigma_{ww}\Pi_w + \Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_w'\Sigma_{ww},
\end{aligned}
$$

which can be estimated by

$$
\frac{1}{n}W'W - \left(\frac{1}{n}W'Y_2\right)\left(\frac{1}{n}Y_2'P_ZY_2\right)^{-1}\left(\frac{1}{n}Y_2'W\right). \blacksquare
$$

B.2 Proof of Proposition 1

When the β is exactly identified by W, then we have

$$
\begin{aligned}
&Y_2'P_WY_2 - \left(Y_2'P_WY_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'P_WY_2\right) \\
&= Y_2'W\left(W'W\right)^{-1}\left[W'W\right]\left(W'W\right)^{-1}W'Y_2 - Y_2'W\left(W'W\right)^{-1}\left[W'Y_2\left(Y_2'P_ZY_2\right)^{-1}Y_2'W\right]\left(W'W\right)^{-1}W'Y_2 \\
&= Y_2'W\left(W'W\right)^{-1}\left[W'W - \left(W'Y_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'W\right)\right]\left(W'W\right)^{-1}W'Y_2
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&P_WY_2\left(Y_2'P_WY_2 - \left(Y_2'P_WY_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'P_WY_2\right)\right)^{-1}Y_2'P_W \\
&\quad= P_WY_2\left(W'Y_2\right)^{-1}W'W\left[W'W - \left(W'Y_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'W\right)\right]^{-1}\left(W'W\right)^{-1}\left(Y_2'W\right)^{-1}Y_2'P_W \\
&\quad= W\left(W'W\right)^{-1}W'Y_2\left(W'Y_2\right)^{-1}W'W\left[W'W - \left(W'Y_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'W\right)\right]^{-1} \\
&\qquad\times\left(W'W\right)^{-1}\left(Y_2'W\right)^{-1}Y_2'W\left(W'W\right)^{-1}W' \\
&\quad= W\left[W'W - \left(W'Y_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'W\right)\right]^{-1}W'
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\mathcal{H}\left(\widehat{\sigma}_\varepsilon^2\right) &= \frac{1}{\widehat{\sigma}_\varepsilon^2}\left(y_1 - Y_2\widehat{\beta}_z\right)'W\left(W'W - \left(W'Y_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'W\right)\right)^{-1}W'\left(y_1 - Y_2\widehat{\beta}_z\right) \\
&= \frac{1}{\widehat{\sigma}_\varepsilon^2}\left(y_1 - Y_2\widehat{\beta}_z\right)'P_WY_2\left[Y_2'P_WY_2 - \left(Y_2'P_WY_2\right)\left(Y_2'P_ZY_2\right)^{-1}\left(Y_2'P_WY_2\right)\right]^{-1}Y_2'P_W\left(y_1 - Y_2\widehat{\beta}_z\right).
\end{aligned}
$$

However, because

$$\begin{aligned} Y_2' P_W \left(y_1 - Y_2 \widehat{\beta}_z \right) &= Y_2' P_W y_1 - Y_2' P_W Y_2 \widehat{\beta}_z \\ &= \left(Y_2' P_W Y_2 \widehat{\beta} \right) \left(\left(Y_2' P_W Y_2 \widehat{\beta} \right)^{-1} \left(Y_2' P_W y_1 \right) - \widehat{\beta}_z \right) \\ &= \widehat{\beta}_w - \widehat{\beta}_z, \end{aligned}$$

we can further write

$$\begin{aligned} \mathcal{H}\left(\widehat{\sigma}_\varepsilon^2\right) &= \frac{1}{\widehat{\sigma}_\varepsilon^2} \left(\widehat{\beta}_w - \widehat{\beta}_z \right)' \left(Y_2' P_W Y_2 \right) \left[Y_2' P_W Y_2 - \left(Y_2' P_W Y_2 \right) \left(Y_2' P_Z Y_2 \right)^{-1} \left(Y_2' P_W Y_2 \right) \right]^{-1} \left(Y_2' P_W Y_2 \right) \left(\widehat{\beta}_w - \widehat{\beta}_z \right) \\ &= \frac{1}{\widehat{\sigma}_\varepsilon^2} \left(\widehat{\beta}_w - \widehat{\beta}_z \right)' \left[\left(Y_2' P_W Y_2 \right)^{-1} - \left(Y_2' P_Z Y_2 \right)^{-1} \right]^{-1} \left(\widehat{\beta}_w - \widehat{\beta}_z \right), \end{aligned}$$

which proves the theorem. ■

C Omitted Proofs of Section 6

C.1 When the Weak IV are Valid Only Under the Alternative Hypothesis

We have

$$y_1 = Y_2\beta + \varepsilon$$

and

$$Y_2 = W\frac{C}{\sqrt{n}} + S\Pi_s + V,$$

where

$$\varepsilon = W\rho_w + V\rho_v + e.$$

The test we consider is the conventional Hausman test:

$$\frac{1}{\hat{\sigma}^2_{\varepsilon,z}}\left(\hat{\beta}_z - \hat{\beta}_s\right)'\left(\left(Y_2'P_sY_2\right)^{-1} - \left(Y_2'P_zY_2\right)^{-1}\right)^{-1}\left(\hat{\beta}_z - \hat{\beta}_s\right).$$

For simplicity, we will assume that the projections w and s are already orthogonal, i.e., $\Gamma_s = 0$. Our simpler model is then written as

$$y_{1i} = Y_{2i}\beta + \varepsilon_i$$

and

$$Y_{2i} = w_i\frac{C}{\sqrt{n}} + s_i\Pi_s + v_i.$$

We use the following symbols for simplicity:

$$\begin{aligned}
A_1 &= \frac{1}{\sqrt{n}}W'\varepsilon,\\
A_2 &= \frac{1}{\sqrt{n}}S'\varepsilon,\\
A_4 &= \frac{1}{\sqrt{n}}Y_2'W,\\
A_5 &= \left(\frac{1}{n}\sum_i w_i s_i\right)C + \frac{1}{\sqrt{n}}\sum_i s_i v_i,\\
\sigma_{ss} &= \sqrt{n}\left(\frac{1}{n}\sum_i s_i^2 - \Sigma_{ss}\right),\\
\sigma_{ws} &= \frac{1}{\sqrt{n}}W'S.
\end{aligned}$$

We begin by expanding the terms in $\hat{\beta}_z - \beta$.

Denominator We note that

$$\begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{W'Y_2}{\sqrt{n}}\right) \\ \frac{1}{n}S'Y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix} + \frac{1}{\sqrt{n}}\begin{bmatrix} \frac{W'Y_2}{\sqrt{n}} \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{w,s} & \frac{1}{n}S'S \end{bmatrix} = \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix} + \frac{1}{\sqrt{n}}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{w,s} & 0 \end{bmatrix}$$

It follows that

$$\begin{aligned}
&\begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{w,s} & \frac{1}{n}S'S \end{bmatrix}^{-1} \\
&= \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \\
&- \frac{1}{\sqrt{n}}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \\
&+ \frac{1}{n}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \\
&+ o_p\left(\frac{1}{n}\right),
\end{aligned}$$

where $o_p\left(\frac{1}{n}\right)$ holds under both the null and the alternative.

Then, under both the null and the alternative, we have

$$\begin{aligned}
\frac{1}{n}Y_2'P_ZY_2 &= \begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{Y_2'W}{\sqrt{n}}\right) & \frac{1}{n}Y_2'S \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{s,w} & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{W'Y_2}{\sqrt{n}}\right) \\ \frac{1}{n}S'Y_2 \end{bmatrix} \\
&= \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix} \\
&+ \frac{1}{\sqrt{n}}f_1 + \frac{1}{n}f_2 + o_p\left(\frac{1}{n}\right)
\end{aligned}$$

with

$$\begin{aligned}
f_1 &= \begin{bmatrix} \frac{Y_2'W}{\sqrt{n}} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} \frac{W'Y_2}{\sqrt{n}} \\ 0 \end{bmatrix} \\
&- \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{w,s} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix} \\
&= 0
\end{aligned}$$

and

$$f_2 = -\begin{bmatrix} \frac{Y_2'W}{\sqrt{n}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix}$$
$$+ \begin{bmatrix} \frac{Y_2'W}{\sqrt{n}} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} \frac{W'Y_2}{\sqrt{n}} \\ 0 \end{bmatrix}$$
$$- \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} \frac{W'Y_2}{\sqrt{n}} \\ 0 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}$$
$$\times \begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{n}S'Y_2 \end{bmatrix}.$$

We note that

$$\begin{aligned}
f_2 &= -\left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1}\sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right) \\
&+ \left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{W'Y_2}{\sqrt{n}}\right) \\
&- \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1}\sigma_{s,w}\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{W'Y_2}{\sqrt{n}}\right) \\
&+ \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1}\sigma_{s,w}\left(\frac{1}{n}W'W\right)^{-1}\sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right) \\
&= \left(\left(\frac{W'Y_2}{\sqrt{n}}\right) - \sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right)\right)'\left(\frac{1}{n}W'W\right)^{-1}\left(\left(\frac{W'Y_2}{\sqrt{n}}\right) - \sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right)\right),
\end{aligned}$$

from which we obtain

$$\begin{aligned}
&\frac{1}{n}Y_2'P_ZY_2 \\
&= \frac{Y_2'P_sY_2}{n} \\
&+ \frac{1}{n}\left(\left(\frac{W'Y_2}{\sqrt{n}}\right) - \sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right)\right)'\left(\frac{1}{n}W'W\right)^{-1}\left(\left(\frac{W'Y_2}{\sqrt{n}}\right) - \sigma_{w,s}\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{n}S'Y_2\right)\right) \\
&+ o_p\left(\frac{1}{n}\right)
\end{aligned}$$

under both the null and the alternative.

Numerator We have

$$\frac{1}{\sqrt{n}}Y_2'P_Z\varepsilon = \left(\frac{1}{n}Y_2'Z\right)\left(\frac{1}{n}Z'Z\right)^{-1}\left(\frac{1}{\sqrt{n}}Z'\varepsilon\right)$$
$$= \begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{Y_2'W}{\sqrt{n}}\right) & \frac{1}{n}Y_2'S \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{s,w} & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

First notice that the norm of

$$\begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{Y_2'W}{\sqrt{n}}\right) & \frac{1}{n}Y_2'S \end{bmatrix}$$
$$\times \left\{ \begin{array}{l} \begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{s,w} & \frac{1}{n}S'S \end{bmatrix}^{-1} - \frac{1}{\sqrt{n}}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \\ +\frac{1}{n}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1}\begin{bmatrix} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \end{array} \right\}$$
$$\times \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$

is bounded by

$$\left\|\begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{Y_2'W}{\sqrt{n}}\right) & \frac{1}{n}Y_2'S \end{bmatrix}\right\| o_p\left(\frac{1}{n}\right)\left\|\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\right\|$$
$$= O_p(1)\, o_p\left(\frac{1}{n}\right) O_p\left(\sqrt{n}\right) = o_p\left(\frac{1}{\sqrt{n}}\right)$$

under both the null and the alternative. Then, we can write under both the null and the alternative

$$\frac{1}{\sqrt{n}}Y_2'P_Z\varepsilon$$
$$= \begin{bmatrix} \frac{1}{\sqrt{n}}\left(\frac{Y_2'W}{\sqrt{n}}\right) & \frac{1}{n}Y_2'S \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & \frac{1}{\sqrt{n}}\sigma_{w,s} \\ \frac{1}{\sqrt{n}}\sigma_{s,w} & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & \frac{1}{n}Y_2'S \end{bmatrix} \begin{bmatrix} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{bmatrix}^{-1} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \frac{1}{\sqrt{n}}g_1 + \frac{1}{n}g_2 + o_p\left(\frac{1}{n}\right)$$
$$= \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1}A_2 + \frac{1}{\sqrt{n}}g_1 + \frac{1}{n}g_2 + o_p\left(\frac{1}{\sqrt{n}}\right)$$
$$= \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1}A_2 + \frac{1}{\sqrt{n}}g_1 + o_p\left(\frac{1}{\sqrt{n}}\right)$$

where

$$
\begin{aligned}
g_1 &= \left[\begin{array}{cc} \frac{Y_2'W}{\sqrt{n}} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \\
&\quad - \left[\begin{array}{cc} 0 & \frac{1}{n}Y_2'S \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \\
&= \left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1} A_1 - \left[\begin{array}{cc} 0 & \frac{1}{n}Y_2'S\left(\frac{1}{n}S'S\right)^{-1} \end{array}\right] \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{c} \left(\frac{1}{n}W'W\right)^{-1} A_1 \\ \left(\frac{1}{n}S'S\right)^{-1} A_2 \end{array}\right] \\
&= \left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1} A_1 - \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1} \sigma_{s,w} \left(\frac{1}{n}W'W\right)^{-1} A_1 \\
&= \left(\left(\frac{Y_2'W}{\sqrt{n}}\right) - \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1} \sigma_{s,w}\right)\left(\frac{1}{n}W'W\right)^{-1} A_1
\end{aligned}
$$

and

$$
\begin{aligned}
g_2 &= -\left[\begin{array}{cc} \frac{Y_2'W}{\sqrt{n}} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \\
&\quad + \left[\begin{array}{cc} 0 & \frac{1}{n}Y_2'S \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \\
&\quad \times \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{cc} \frac{1}{n}W'W & 0 \\ 0 & \frac{1}{n}S'S \end{array}\right]^{-1} \left[\begin{array}{c} A_1 \\ A_2 \end{array}\right] \\
&= -\left[\begin{array}{cc} \left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1} & 0 \end{array}\right] \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{c} \left(\frac{1}{n}W'W\right)^{-1} A_1 \\ \left(\frac{1}{n}S'S\right)^{-1} A_2 \end{array}\right] \\
&\quad + \left[\begin{array}{cc} 0 & \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1} \end{array}\right] \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{cc} \left(\frac{1}{n}W'W\right)^{-1} & 0 \\ 0 & \left(\frac{1}{n}S'S\right)^{-1} \end{array}\right] \left[\begin{array}{cc} 0 & \sigma_{w,s} \\ \sigma_{s,w} & 0 \end{array}\right] \left[\begin{array}{c} \left(\frac{1}{n}W'W\right)^{-1} A_1 \\ \left(\frac{1}{n}S'S\right)^{-1} A_2 \end{array}\right] \\
&= -\left(\frac{Y_2'W}{\sqrt{n}}\right)\left(\frac{1}{n}W'W\right)^{-1} \sigma_{w,s} \left(\frac{1}{n}S'S\right)^{-1} A_2 \\
&\quad + \left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1} \sigma_{s,w} \left(\frac{1}{n}W'W\right)^{-1} \sigma_{w,s} \left(\frac{1}{n}S'S\right)^{-1} A_2 \\
&= O_p(1) \text{ under both the null and the alternative.}
\end{aligned}
$$

We now go back to the Hausman test.

Numerator We have

$$
\begin{aligned}
&\sqrt{n}\left(\hat{\beta}_z-\beta\right)-\sqrt{n}\left(\hat{\beta}_s-\beta\right)\\
&=\left(\frac{Y_2'P_sY_2}{n}+\frac{1}{n}f_2+o_p\left(\frac{1}{n}\right)\right)^{-1}\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}+\frac{1}{\sqrt{n}}g_1+o_p\left(\frac{1}{\sqrt{n}}\right)\right)\\
&-\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}\right)\\
&=\left(\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}-\frac{1}{n}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}f_2\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}+o_p\left(\frac{1}{n}\right)\right)\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}+\frac{1}{\sqrt{n}}g_1+o_p\left(\frac{1}{\sqrt{n}}\right)\right)\\
&-\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}\right)\\
&=\frac{1}{\sqrt{n}}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}g_1\\
&-\frac{1}{n}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}f_2\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}\right)+o_p\left(\frac{1}{\sqrt{n}}\right)\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}\\
&=\frac{1}{\sqrt{n}}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}g_1-\frac{1}{n}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}f_2\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}\left(\frac{Y_2'P_s\varepsilon}{\sqrt{n}}\right)+o_p\left(\frac{1}{\sqrt{n}}\right)
\end{aligned}
$$

under both the null and the alternative.

Denominator

$$
\begin{aligned}
&\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}-\left(\frac{Y_2'P_zY_2}{n}\right)^{-1}\\
&=\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}-\left(\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}-\frac{1}{n}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}f_2\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}+o_p\left(\frac{1}{n}\right)\right)\\
&=\frac{1}{n}\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}f_2\left(\frac{Y_2'P_sY_2}{n}\right)^{-1}+o_p\left(\frac{1}{n}\right)
\end{aligned}
$$

under both the null and the alternative.

Hausman Test Statistic without the Variance Estimator Therefore, we have

$$
T=\left(\hat{\beta}_z-\hat{\beta}_s\right)'\left(\left(Y_2'P_sY_2\right)^{-1}-\left(Y_2'P_zY_2\right)^{-1}\right)^{-1}\left(\hat{\beta}_z-\hat{\beta}_s\right)=g_1'f_2^{-1}g_1+o_p\left(1\right).
$$

But because

$$
\begin{aligned}
g_1&=\left(\left(\frac{Y_2'W}{\sqrt{n}}\right)-\left(\frac{1}{n}Y_2'S\right)\left(\frac{1}{n}S'S\right)^{-1}\left(\frac{1}{\sqrt{n}}S'W\right)\right)\left(\frac{1}{n}W'W\right)^{-1}A_1\\
&=\frac{1}{\sqrt{n}}\left(Y_2'M_sW\right)\left(\frac{1}{n}W'W\right)^{-1}A_1
\end{aligned}
$$

and

$$
\begin{aligned}
f_2 =& \left(\left(\frac{W'Y_2}{\sqrt{n}} \right) - \left(\frac{1}{\sqrt{n}} W'S \right) \left(\frac{1}{n} S'S \right)^{-1} \left(\frac{1}{n} S'Y_2 \right) \right)' \left(\frac{1}{n} W'W \right)^{-1} \\
&\times \left(\left(\frac{W'Y_2}{\sqrt{n}} \right) - \left(\frac{1}{\sqrt{n}} W'S \right) \left(\frac{1}{n} S'S \right)^{-1} \left(\frac{1}{n} S'Y_2 \right) \right) \\
=& \frac{1}{n} \left(Y_2' M_s W \right) \left(\frac{1}{n} W'W \right)^{-1} \left(W' M_s Y_2 \right)
\end{aligned}
$$

we obtain

$$
\begin{aligned}
T &= A_1' \left(\frac{1}{n} W'W \right)^{-1} A_1 + o_p(1) \\
&= \left(\frac{1}{\sqrt{n}} \varepsilon' W \right) \left(\frac{1}{n} W'W \right)^{-1} \left(\frac{1}{\sqrt{n}} W' \varepsilon \right) + o_p(1)
\end{aligned}
$$

and

$$
\mathcal{H} = \frac{T}{\widehat{\sigma}_{\varepsilon}^2} = \frac{\left(\frac{1}{\sqrt{n}} \varepsilon' W \right) \left(\frac{1}{n} W'W \right)^{-1} \left(\frac{1}{\sqrt{n}} W' \varepsilon \right)}{\sigma_{\varepsilon}^2} + o_p(1)
$$

under both the null and the alternative, which yields the desired results.

C.2 Proof of (12)

Assume that we adopt the Staiger-Stock asymptotics. Because the 2SLS $\widehat{\beta}_z$ is consistent for $\beta + \tau$ under Staiger-Stock asymptotics with heteroscedasticity, we may write

$$
\hat{\varepsilon}_z = \varepsilon - Y_2 \tau - Y_2 \left(\widehat{\beta}_z - \beta - \tau \right),
$$

and we may easily deduce that

$$\frac{\left(\widehat{\beta}_w-\widehat{\beta}_z\right)^2}{\widehat{\mathrm{Var}}\left(\widehat{\beta}_w-\widehat{\beta}_z\right)}$$

$$= \left[\begin{array}{c} \frac{\left(\frac{1}{n}\sum_{i=1}^n w_i^2(\varepsilon_i-Y_{2i}\tau)^2\right)}{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)^2} - 2\frac{\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iw_i(\varepsilon_i-Y_{2i}\tau)^2\right)}{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]} \\ +\frac{\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'(\varepsilon_i-Y_{2i}\tau)^2\right)\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]^2} \end{array}\right]^{-1}$$

$$\times\left[\frac{\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i\varepsilon_i}{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)} - \frac{\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n z_i\varepsilon_i\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]}\right]^2 + o_p(1)$$

$$= \left[\begin{array}{c} \left(\frac{1}{n}\sum_{i=1}^n w_i^2\left(\varepsilon_i-Y_{2i}\tau\right)^2\right) - \frac{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iw_i(\varepsilon_i-Y_{2i}\tau)^2\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]} \\ +\frac{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)^2\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'(\varepsilon_i-Y_{2i}\tau)^2\right)\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]^2} \end{array}\right]^{-1}$$

$$\times\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i\varepsilon_i - \frac{\left(\frac{1}{n}\sum_{i=1}^n w_iY_{2i}\right)\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n z_i\varepsilon_i\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]}\right]^2 + o_p(1).$$

Notice that under the Staiger-Stock asymptotics, $\frac{1}{\sqrt{n}}\sum_{i=1}^n w_iY_{2i} = O_p(1)$ under the both null and alternative. Then, one might expect that

$$\frac{\left(\widehat{\beta}_w-\widehat{\beta}_z\right)^2}{\widehat{\mathrm{Var}}\left(\widehat{\beta}_w-\widehat{\beta}_z\right)}$$

$$= \frac{\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i\varepsilon_i - \frac{\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n w_iY_{2i}\right)\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_i\varepsilon_i\right)}{\left[\left(\frac{1}{n}\sum_{i=1}^n Y_{2i}z_i'\right)\left(\frac{1}{n}\sum_{i=1}^n z_iz_i'\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^n z_iY_{2i}\right)\right]}\right]^2}{\left(\frac{1}{n}\sum_{i=1}^n w_i^2\left(\varepsilon_i-Y_{2i}\tau\right)^2\right)} + o_p(1)$$

$$= \frac{\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i\varepsilon_i - \left(\frac{1}{\sqrt{n}}\sum_{i=1}^n w_iY_{2i}\right)\left(\frac{\Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\rho_s}{\Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s}\right)\right]^2}{\left(\frac{1}{n}\sum_{i=1}^n w_i^2\left(\varepsilon_i-Y_{2i}\tau\right)^2\right)} + o_p(1)$$

$$= \frac{\left[\frac{1}{\sqrt{n}}\sum_{i=1}^n w_i\left(\varepsilon_i-Y_{2i}\tau\right)\right]^2}{\left(\frac{1}{n}\sum_{i=1}^n w_i^2\left(\varepsilon_i-Y_{2i}\tau\right)^2\right)} + o_p(1).$$

Under the alternative, since $\varepsilon - Y_2\tau = \widetilde{S}\left(\rho_s - \Pi_s\tau\right) + V\left(\rho_v - \tau\right) + e - W\frac{C}{\sqrt{n}}\tau$, we have

$$
\begin{aligned}
&\frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i\left(\varepsilon_i - Y_{2i}\tau\right) \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n} w_i\left(\tilde{s}_i\left(\rho_s - \Pi_s\tau\right) + v_i\left(\rho_v - \tau\right) + e_i\right) - \frac{1}{n}\sum_{i=1}^{n} w_i^2 C\tau \\
&\Rightarrow N\left(-\Sigma_{ww}C\tau, \lim_n \frac{1}{n}\sum_{i=1}^{n} E\left[w_i^2\left(\tilde{s}_i\left(\rho_s - \Pi_s\tau\right) + v_i\left(\rho_v - \tau\right) + e_i\right)^2\right]\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{n}\sum_{i=1}^{n} w_i^2\left(\varepsilon_i - Y_{2i}\tau\right)^2 \\
&= \lim_n \frac{1}{n}\sum_{i=1}^{n} E\left[w_i^2\left(\tilde{s}_i\left(\rho_s - \Pi_s\tau\right) + v_i\left(\rho_v - \tau\right) + e_i\right)^2\right] + o\left(1\right).
\end{aligned}
$$

In this case

$$
\frac{\left(\widehat{\beta}_w - \widehat{\beta}_z\right)^2}{\widehat{\operatorname{Var}}\left(\widehat{\beta}_w - \widehat{\beta}_z\right)} \Rightarrow \left(\mathcal{Z} + \kappa_{hetero}\right)^2 \equiv \chi_1^2\left(\kappa_{hetero}\right),
$$

where

$$
\kappa_{hetero} = \frac{-\Sigma_{ww}C\tau}{\lim_n \frac{1}{n}\sum_{i=1}^{n} E\left[w_i^2\left(\tilde{s}_i\left(\rho_s - \Pi_s\tau\right) + v_i\left(\rho_v - \tau\right) + e_i\right)^2\right]},
$$

and the test becomes asymptotically unbiased. ■

D Proofs for Appendix B

D.1 Proof of Lemma 2

Part (a): The result follows because

$$\begin{aligned}\frac{1}{n}Z'Z &= \begin{bmatrix} \frac{1}{n}W'W & \frac{1}{n}W'S \\ \frac{1}{n}S'W & \frac{1}{n}S'S \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{n}W'W & \frac{1}{n}W'W\Gamma_w + \frac{1}{n}W'\widetilde{S} \\ \frac{1}{n}\widetilde{S}'W + \frac{1}{n}\Gamma_w'W'W & \frac{1}{n}\widetilde{S}'\widetilde{S} + \frac{1}{n}\widetilde{S}'W\Gamma_w + \frac{1}{n}\Gamma_w'W'\widetilde{S} + \frac{1}{n}\Gamma_w'W'W\Gamma_w \end{bmatrix}\end{aligned}$$

and by the WLLN. ■

Part (b): By definition,

$$\frac{1}{n}Z'Y_2 = \begin{bmatrix} \frac{1}{n}W'Y_2 \\ \frac{1}{n}\Gamma_w'W'Y_2 + \frac{1}{n}\widetilde{S}'Y_2 \end{bmatrix}.$$

Notice that

$$\frac{1}{n}W'Y_2 = \frac{1}{n}W'W\frac{C}{\sqrt{n}} + \frac{1}{n}W'\widetilde{S}\Pi_s + \frac{1}{n}W'V = o_p(1).$$

Also,

$$\frac{1}{n}\widetilde{S}'Y_2 = \frac{1}{n}\widetilde{S}'W\frac{C}{\sqrt{n}} + \frac{1}{n}\widetilde{S}'\widetilde{S}\Pi_s + \frac{1}{n}\widetilde{S}'V = \frac{1}{n}\widetilde{S}'\widetilde{S}\Pi_s + o_p(1).$$

Then the required result follows by the WLLN. ■

Part (c): The required result follows by Parts (a), (b) and the Slutzky theorem. ■

D.2 Proof of Lemma 3

Part (a): Since $\rho_s = 0$ under the null,

$$\frac{1}{\sqrt{n}}W'\varepsilon = \frac{1}{\sqrt{n}}W'V\rho_v + \frac{1}{\sqrt{n}}W'e.$$

The required result follows by the CLT assumption in Condition 2. ■

Part (b): By definition

$$\begin{bmatrix} Y_2'P_WY_2 \\ Y_2'P_W\varepsilon \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{n}W'WC + \frac{1}{\sqrt{n}}W'\widetilde{S}\Pi_s + \frac{1}{\sqrt{n}}W'V\right)'\left(\frac{1}{n}W'W\right)^{-1}\left(\frac{1}{n}W'WC + \frac{1}{\sqrt{n}}W'\widetilde{S}\Pi_s + \frac{1}{\sqrt{n}}W'V\right) \\ \left(\frac{1}{n}W'WC + \frac{1}{\sqrt{n}}W'\widetilde{S}\Pi_s + \frac{1}{\sqrt{n}}W'V\right)'\left(\frac{1}{n}W'W\right)^{-1}\frac{1}{\sqrt{n}}W'\varepsilon \end{bmatrix}.$$

The required result follows by Conditions 1 and 2, Part (a), and the generalized Slutzky theorem. ■

Part (c): Under the null, $E(W'\varepsilon) = E(S'\varepsilon) = 0$. The required result follows by the WLLN assumption in Condition 1(ii) and the fact that $\varepsilon = V\rho_v + e$ under the null. ■

Part (d): By definition,

$$\frac{1}{n}Y_2'\varepsilon = \frac{1}{n}\left(W\frac{C}{\sqrt{n}} + \widetilde{S}\Pi_s\right)'\varepsilon + \frac{1}{n}V'\varepsilon = o_p(1) + \frac{1}{n}V'V\rho_v + \frac{1}{n}V'e = \frac{1}{n}V'V\rho_v + o_p(1).$$

The required result follows by the WLLN in Condition 1(ii) and the fact that $\varepsilon = V\rho_v + e$ under the null. ■

Part (e): The required result follows by Lemma 4(b) with $\tau = 0$ under the null due to $\rho_s = 0$. ■

Part (f): The required result follows by Part (e) and the usual arguments. ■

Part (g): The required result follows by Lemma 4(c) with $\tau = 0$, $\rho_s = 0$, and $\varepsilon = e$ under the null. ■

D.3 Proof of Lemma 4

Part (a): The required result follows by Conditions 1 because

$$\frac{1}{n}Z'\varepsilon = \begin{bmatrix} \frac{1}{n}W'\varepsilon \\ \frac{1}{n}\left(W\Gamma_w + \widetilde{S}\right)'\left(\widetilde{S}\rho_s + V\rho_v + e\right) \end{bmatrix} = \begin{bmatrix} o_p(1) \\ \frac{1}{n}\widetilde{S}'\widetilde{S}\rho_s + o_p(1) \end{bmatrix}. \ ■$$

Part (b): By definition

$$\widehat{\beta}_z - \beta = \left(Y_2'P_ZY_2\right)^{-1}Y_2'P_Z\varepsilon \to_p \left(\Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\Pi_s\right)^{-1}\Pi_s'\Sigma_{\widetilde{s}\widetilde{s}}\rho_s,$$

where the last convergence follows by Lemma 2(a), (b) and Lemma 4(a). ■

Part (c): By definition,

$$\varepsilon - Y_2\tau = \widetilde{S}\left(\rho_s - \Pi_s\tau\right) + V\left(\rho_v - \tau\right) + e - W\frac{C}{\sqrt{n}}\tau.$$

Then

$$\frac{1}{\sqrt{n}}W'\left(\varepsilon - Y_2\tau\right) = \frac{1}{\sqrt{n}}W'\widetilde{S}\left(\rho_s - \Pi_s\tau\right) - \frac{1}{n}W'WC\tau + \frac{1}{\sqrt{n}}W'V\left(\rho_v - \tau\right) + \frac{1}{\sqrt{n}}W'e.$$

The required result follows by the Conditions 1 and 2. ■

Part (d): By definition,

$$\begin{aligned}
\widetilde{\sigma}^2_{\varepsilon,z} &= \frac{1}{n}\widehat{\varepsilon}_z'M_Z\widehat{\varepsilon}_z = \frac{1}{n}\left(y_1 - Y_2\widehat{\beta}_z\right)'M_Z\left(y_1 - Y_2\widehat{\beta}_z\right) \\
&= \frac{1}{n}\left(\varepsilon - Y_2\tau - Y_2\left(\widehat{\beta}_z - \beta - \tau\right)\right)'M_Z\left(\varepsilon - Y_2\tau - Y_2\left(\widehat{\beta}_z - \beta - \tau\right)\right) \\
&= \frac{1}{n}\left(\varepsilon - Y_2\tau\right)'M_Z\left(\varepsilon - Y_2\tau\right) + o_p(1),
\end{aligned}$$

where the third equality follows by Part (b), $\widehat{\beta}_z - \beta - \tau = o_p(1)$, and $\frac{1}{n}\left(\varepsilon - Y_2\tau\right)'\left(\varepsilon - Y_2\tau\right) = O_p(1)$ and $\frac{1}{n}Y_2'Y_2 = O_p(1)$ by Condition 1. Because

$$\begin{aligned}
\varepsilon - Y_2\tau &= \widetilde{S}\rho_s + V\rho_v + e - \left(W\frac{C}{\sqrt{n}} + \widetilde{S}\Pi_s + V\right)\tau \\
&= \widetilde{S}\left(\rho_s - \Pi_s\tau\right) + V\left(\rho_v - \tau\right) + e - \frac{1}{\sqrt{n}}WC\tau
\end{aligned}$$

we can further write

$$M_Z\left(\varepsilon - Y_2\tau\right) = M_ZV\left(\rho_v - \tau\right) + M_Ze$$

and therefore,

$$\begin{aligned}\widetilde{\sigma}^2_{\varepsilon,z} &= \frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)' M_Z \left(V\left(\rho_v-\tau\right)+e\right)+o_p\left(1\right)\\ &= \frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)'\left(V\left(\rho_v-\tau\right)+e\right)\\ &\quad -\left[\frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)' Z\right]\left(\frac{1}{n}Z'Z\right)^{-1}\left[\frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)' Z\right]+o_p\left(1\right).\end{aligned}$$

Notice that under Conditions 1,

$$\frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)'\left(V\left(\rho_v-\tau\right)+e\right)\rightarrow_p \left(\rho_v-\tau\right)'\Sigma_{vv}\left(\rho_v-\tau\right)+\sigma_e^2=\sigma_*^2.$$

$$\frac{1}{n}\left(V\left(\rho_v-\tau\right)+e\right)' Z\rightarrow_p 0$$

$$\left(\frac{1}{n}Z'Z\right)^{-1}\rightarrow_p \begin{bmatrix}\Sigma_{ww} & \Sigma_{ww}\Gamma_w\\ \Gamma_w'\Sigma_{ww} & \Gamma_w'\Sigma_{ww}\Gamma_w+\Sigma_{\widetilde{s}\widetilde{s}}\end{bmatrix}^{-1}>0,$$

where the second convergence holds because $E\left[v_i z_i\right]=0$ and $E\left[e_i z_i\right]=0$ and the third convergence holds by Lemma 2(a). Therefore we have the required result

$$\widetilde{\sigma}^2_{\varepsilon,z}\rightarrow_p \left(\rho_v-\tau\right)'\Sigma_{vv}\left(\rho_v-\tau\right)+\sigma_e^2. \blacksquare$$

Part (e): By definition,

$$\begin{aligned}\widehat{\sigma}^2_{\varepsilon,z} &= \frac{1}{n}\left(y_1-Y_2\widehat{\beta}_z\right)'\left(y_1-Y_2\widehat{\beta}_z\right)\\ &= \frac{1}{n}\left(\varepsilon-Y_2\tau-Y_2\left(\widehat{\beta}_z-\beta-\tau\right)\right)'\left(\varepsilon-Y_2\tau-Y_2\left(\widehat{\beta}_z-\beta-\tau\right)\right)\\ &= \frac{1}{n}\left(\varepsilon-Y_2\tau\right)'\left(\varepsilon-Y_2\tau\right)+o_p\left(1\right)\\ &= \frac{1}{n}\left(\widetilde{S}\left(\rho_s-\Pi_s\tau\right)+V\left(\rho_v-\tau\right)+e-W\frac{C}{\sqrt{n}}\tau\right)'\left(\widetilde{S}\left(\rho_s-\Pi_s\tau\right)+V\left(\rho_v-\tau\right)+e-W\frac{C}{\sqrt{n}}\tau\right)+o_p\left(1\right)\\ &\rightarrow_p \left(\rho_s-\Pi_s\tau\right)'\Sigma_{\widetilde{s}\widetilde{s}}\left(\rho_s-\Pi_s\tau\right)+\left(\rho_v-\tau\right)'\Sigma_{vv}\left(\rho_v-\tau\right)+\sigma_e^2,\end{aligned}$$

where the third equality follows by Part (b), $\widehat{\beta}_z-\beta-\tau=o_p\left(1\right).$ $\blacksquare$

Table 1: Size, ρ=.25

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	1	1	0.01	0.00	0.00	0.05	0.05
100	1	1	0.02	0.01	0.01	0.05	0.06
100	1	1	0.03	0.01	0.01	0.05	0.06
100	1	1	0.05	0.01	0.01	0.05	0.06
100	1	1	0.1	0.02	0.02	0.05	0.06
100	1	1	0.2	0.03	0.04	0.05	0.05
200	1	1	0.01	0.01	0.01	0.05	0.05
200	1	1	0.02	0.01	0.01	0.05	0.05
200	1	1	0.03	0.01	0.01	0.05	0.05
200	1	1	0.05	0.02	0.02	0.05	0.05
200	1	1	0.1	0.03	0.03	0.05	0.05
200	1	1	0.2	0.04	0.04	0.05	0.05
500	1	1	0.01	0.01	0.01	0.05	0.05
500	1	1	0.02	0.02	0.02	0.05	0.05
500	1	1	0.03	0.03	0.03	0.05	0.05
500	1	1	0.05	0.03	0.03	0.05	0.06
500	1	1	0.1	0.04	0.04	0.05	0.05
500	1	1	0.2	0.05	0.05	0.06	0.06
100	1	2	0.01	0.00	0.00	0.05	0.06
100	1	2	0.02	0.01	0.01	0.05	0.06
100	1	2	0.03	0.01	0.01	0.05	0.05
100	1	2	0.05	0.01	0.01	0.05	0.06
100	1	2	0.1	0.02	0.02	0.05	0.06
100	1	2	0.2	0.03	0.03	0.06	0.06
200	1	2	0.01	0.01	0.01	0.05	0.06
200	1	2	0.02	0.01	0.01	0.05	0.06
200	1	2	0.03	0.01	0.01	0.05	0.05
200	1	2	0.05	0.02	0.02	0.05	0.05
200	1	2	0.1	0.03	0.03	0.05	0.05
200	1	2	0.2	0.04	0.04	0.05	0.05
500	1	2	0.01	0.01	0.01	0.06	0.06
500	1	2	0.02	0.02	0.02	0.06	0.06
500	1	2	0.03	0.03	0.03	0.06	0.06
500	1	2	0.05	0.04	0.04	0.06	0.06
500	1	2	0.1	0.04	0.05	0.05	0.06
500	1	2	0.2	0.05	0.05	0.06	0.06
100	1	5	0.01	0.00	0.00	0.05	0.05
100	1	5	0.02	0.00	0.00	0.05	0.06
100	1	5	0.03	0.01	0.01	0.05	0.06
100	1	5	0.05	0.01	0.01	0.05	0.06
100	1	5	0.1	0.02	0.02	0.05	0.06
100	1	5	0.2	0.03	0.03	0.05	0.06
200	1	5	0.01	0.01	0.01	0.05	0.06
200	1	5	0.02	0.01	0.01	0.05	0.06
200	1	5	0.03	0.01	0.01	0.05	0.06
200	1	5	0.05	0.02	0.02	0.05	0.06
200	1	5	0.1	0.03	0.03	0.06	0.06
200	1	5	0.2	0.04	0.05	0.06	0.06
500	1	5	0.01	0.01	0.01	0.05	0.05
500	1	5	0.02	0.02	0.02	0.05	0.05
500	1	5	0.03	0.02	0.02	0.05	0.05
500	1	5	0.05	0.03	0.03	0.05	0.05
500	1	5	0.1	0.04	0.04	0.05	0.05
500	1	5	0.2	0.04	0.04	0.05	0.05

Table 2: Size, ρ=.5

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	1	1	0.01	0.02	0.02	0.05	0.06
100	1	1	0.02	0.03	0.02	0.05	0.06
100	1	1	0.03	0.03	0.03	0.05	0.06
100	1	1	0.05	0.04	0.03	0.05	0.06
100	1	1	0.1	0.05	0.04	0.05	0.06
100	1	1	0.2	0.06	0.04	0.05	0.06
200	1	1	0.01	0.02	0.02	0.05	0.05
200	1	1	0.02	0.03	0.03	0.05	0.05
200	1	1	0.03	0.04	0.03	0.05	0.05
200	1	1	0.05	0.04	0.03	0.05	0.05
200	1	1	0.1	0.05	0.04	0.05	0.05
200	1	1	0.2	0.05	0.04	0.05	0.05
500	1	1	0.01	0.04	0.03	0.05	0.05
500	1	1	0.02	0.04	0.04	0.05	0.05
500	1	1	0.03	0.04	0.04	0.05	0.06
500	1	1	0.05	0.05	0.04	0.06	0.06
500	1	1	0.1	0.05	0.05	0.05	0.05
500	1	1	0.2	0.05	0.05	0.06	0.06
100	1	2	0.01	0.01	0.01	0.05	0.06
100	1	2	0.02	0.02	0.02	0.05	0.06
100	1	2	0.03	0.02	0.02	0.05	0.06
100	1	2	0.05	0.03	0.02	0.05	0.06
100	1	2	0.1	0.04	0.03	0.06	0.06
100	1	2	0.2	0.04	0.03	0.06	0.06
200	1	2	0.01	0.02	0.02	0.06	0.06
200	1	2	0.02	0.03	0.03	0.06	0.06
200	1	2	0.03	0.04	0.03	0.05	0.06
200	1	2	0.05	0.04	0.04	0.05	0.06
200	1	2	0.1	0.05	0.04	0.05	0.06
200	1	2	0.2	0.05	0.04	0.05	0.06
500	1	2	0.01	0.03	0.03	0.06	0.06
500	1	2	0.02	0.04	0.04	0.06	0.06
500	1	2	0.03	0.05	0.04	0.06	0.06
500	1	2	0.05	0.05	0.04	0.06	0.06
500	1	2	0.1	0.05	0.05	0.06	0.06
500	1	2	0.2	0.05	0.05	0.06	0.06
100	1	5	0.01	0.01	0.01	0.05	0.06
100	1	5	0.02	0.02	0.01	0.05	0.06
100	1	5	0.03	0.02	0.01	0.05	0.06
100	1	5	0.05	0.02	0.02	0.05	0.06
100	1	5	0.1	0.02	0.02	0.06	0.07
100	1	5	0.2	0.03	0.03	0.06	0.07
200	1	5	0.01	0.02	0.02	0.05	0.06
200	1	5	0.02	0.03	0.02	0.06	0.06
200	1	5	0.03	0.03	0.03	0.06	0.06
200	1	5	0.05	0.03	0.03	0.06	0.06
200	1	5	0.1	0.04	0.03	0.06	0.07
200	1	5	0.2	0.04	0.04	0.06	0.07
500	1	5	0.01	0.03	0.03	0.05	0.05
500	1	5	0.02	0.03	0.03	0.05	0.05
500	1	5	0.03	0.03	0.03	0.05	0.05
500	1	5	0.05	0.04	0.03	0.05	0.05
500	1	5	0.1	0.04	0.04	0.05	0.05
500	1	5	0.2	0.04	0.04	0.05	0.05

Table 3: Size, ρ=.75

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	1	1	0.01	0.07	0.06	0.06	0.07
100	1	1	0.02	0.08	0.06	0.06	0.07
100	1	1	0.03	0.08	0.06	0.06	0.07
100	1	1	0.05	0.08	0.06	0.06	0.07
100	1	1	0.1	0.08	0.05	0.06	0.06
100	1	1	0.2	0.08	0.05	0.06	0.06
200	1	1	0.01	0.07	0.06	0.05	0.06
200	1	1	0.02	0.07	0.06	0.05	0.05
200	1	1	0.03	0.08	0.06	0.05	0.06
200	1	1	0.05	0.07	0.06	0.05	0.05
200	1	1	0.1	0.07	0.05	0.05	0.05
200	1	1	0.2	0.07	0.04	0.05	0.05
500	1	1	0.01	0.07	0.07	0.05	0.06
500	1	1	0.02	0.07	0.06	0.06	0.06
500	1	1	0.03	0.06	0.06	0.06	0.06
500	1	1	0.05	0.06	0.05	0.06	0.06
500	1	1	0.1	0.06	0.05	0.05	0.05
500	1	1	0.2	0.06	0.05	0.06	0.06
100	1	2	0.01	0.06	0.05	0.06	0.07
100	1	2	0.02	0.07	0.05	0.06	0.06
100	1	2	0.03	0.07	0.05	0.06	0.07
100	1	2	0.05	0.07	0.05	0.06	0.06
100	1	2	0.1	0.07	0.04	0.06	0.06
100	1	2	0.2	0.06	0.03	0.06	0.07
200	1	2	0.01	0.08	0.07	0.06	0.06
200	1	2	0.02	0.08	0.07	0.06	0.06
200	1	2	0.03	0.08	0.06	0.06	0.06
200	1	2	0.05	0.08	0.06	0.06	0.06
200	1	2	0.1	0.06	0.04	0.05	0.06
200	1	2	0.2	0.06	0.04	0.05	0.06
500	1	2	0.01	0.08	0.07	0.06	0.06
500	1	2	0.02	0.07	0.06	0.06	0.06
500	1	2	0.03	0.07	0.06	0.06	0.06
500	1	2	0.05	0.06	0.05	0.06	0.06
500	1	2	0.1	0.06	0.05	0.06	0.06
500	1	2	0.2	0.06	0.05	0.06	0.06
100	1	5	0.01	0.04	0.03	0.06	0.07
100	1	5	0.02	0.04	0.03	0.06	0.07
100	1	5	0.03	0.04	0.03	0.06	0.07
100	1	5	0.05	0.04	0.03	0.06	0.08
100	1	5	0.1	0.04	0.02	0.07	0.08
100	1	5	0.2	0.03	0.02	0.07	0.09
200	1	5	0.01	0.06	0.05	0.06	0.06
200	1	5	0.02	0.06	0.05	0.06	0.06
200	1	5	0.03	0.06	0.05	0.06	0.06
200	1	5	0.05	0.05	0.04	0.06	0.07
200	1	5	0.1	0.04	0.03	0.06	0.07
200	1	5	0.2	0.04	0.04	0.07	0.08
500	1	5	0.01	0.06	0.05	0.05	0.05
500	1	5	0.02	0.05	0.05	0.05	0.05
500	1	5	0.03	0.05	0.04	0.05	0.05
500	1	5	0.05	0.04	0.04	0.05	0.05
500	1	5	0.1	0.04	0.04	0.05	0.05
500	1	5	0.2	0.04	0.04	0.05	0.05

Table 4: Size, ρ=.25

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	2	1	0.01	0.01	0.01	0.05	0.06
100	2	1	0.02	0.01	0.01	0.05	0.06
100	2	1	0.03	0.01	0.01	0.05	0.06
100	2	1	0.05	0.01	0.02	0.05	0.06
100	2	1	0.1	0.02	0.02	0.05	0.06
100	2	1	0.2	0.03	0.03	0.05	0.06
200	2	1	0.01	0.01	0.01	0.05	0.06
200	2	1	0.02	0.01	0.01	0.05	0.06
200	2	1	0.03	0.02	0.02	0.05	0.06
200	2	1	0.05	0.02	0.02	0.05	0.06
200	2	1	0.1	0.03	0.03	0.05	0.06
200	2	1	0.2	0.04	0.04	0.05	0.06
500	2	1	0.01	0.02	0.02	0.05	0.06
500	2	1	0.02	0.03	0.03	0.06	0.06
500	2	1	0.03	0.03	0.03	0.06	0.06
500	2	1	0.05	0.04	0.04	0.06	0.06
500	2	1	0.1	0.05	0.05	0.05	0.05
500	2	1	0.2	0.05	0.05	0.05	0.06
100	2	2	0.01	0.01	0.01	0.05	0.06
100	2	2	0.02	0.01	0.01	0.05	0.06
100	2	2	0.03	0.01	0.01	0.05	0.05
100	2	2	0.05	0.02	0.02	0.05	0.05
100	2	2	0.1	0.03	0.03	0.05	0.05
100	2	2	0.2	0.03	0.04	0.05	0.06
200	2	2	0.01	0.01	0.01	0.05	0.05
200	2	2	0.02	0.01	0.01	0.05	0.05
200	2	2	0.03	0.02	0.02	0.05	0.05
200	2	2	0.05	0.03	0.03	0.05	0.05
200	2	2	0.1	0.04	0.04	0.05	0.05
200	2	2	0.2	0.04	0.04	0.05	0.05
500	2	2	0.01	0.02	0.02	0.06	0.06
500	2	2	0.02	0.02	0.03	0.06	0.06
500	2	2	0.03	0.03	0.03	0.05	0.06
500	2	2	0.05	0.04	0.04	0.05	0.06
500	2	2	0.1	0.04	0.05	0.05	0.06
500	2	2	0.2	0.05	0.05	0.06	0.06
100	2	5	0.01	0.01	0.01	0.06	0.06
100	2	5	0.02	0.01	0.01	0.06	0.06
100	2	5	0.03	0.01	0.01	0.05	0.07
100	2	5	0.05	0.01	0.02	0.05	0.06
100	2	5	0.1	0.02	0.03	0.05	0.06
100	2	5	0.2	0.03	0.04	0.05	0.07
200	2	5	0.01	0.01	0.01	0.06	0.06
200	2	5	0.02	0.01	0.01	0.06	0.06
200	2	5	0.03	0.01	0.02	0.06	0.06
200	2	5	0.05	0.02	0.02	0.06	0.06
200	2	5	0.1	0.03	0.03	0.06	0.06
200	2	5	0.2	0.04	0.05	0.06	0.06
500	2	5	0.01	0.01	0.01	0.05	0.05
500	2	5	0.02	0.02	0.02	0.05	0.05
500	2	5	0.03	0.02	0.02	0.05	0.05
500	2	5	0.05	0.03	0.03	0.05	0.05
500	2	5	0.1	0.04	0.04	0.05	0.05
500	2	5	0.2	0.04	0.04	0.05	0.05

Table 4 - Continued: Size, ρ=.25

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	5	1	0.01	0.02	0.02	0.06	0.07
100	5	1	0.02	0.02	0.03	0.06	0.07
100	5	1	0.03	0.02	0.03	0.06	0.07
100	5	1	0.05	0.03	0.03	0.06	0.07
100	5	1	0.1	0.04	0.04	0.05	0.07
100	5	1	0.2	0.04	0.04	0.05	0.07
200	5	1	0.01	0.03	0.03	0.06	0.07
200	5	1	0.02	0.03	0.03	0.06	0.07
200	5	1	0.03	0.03	0.04	0.06	0.07
200	5	1	0.05	0.04	0.04	0.06	0.07
200	5	1	0.1	0.04	0.04	0.05	0.07
200	5	1	0.2	0.05	0.05	0.05	0.07
500	5	1	0.01	0.03	0.03	0.06	0.06
500	5	1	0.02	0.04	0.04	0.05	0.06
500	5	1	0.03	0.04	0.04	0.05	0.06
500	5	1	0.05	0.04	0.04	0.05	0.06
500	5	1	0.1	0.04	0.05	0.05	0.06
500	5	1	0.2	0.05	0.05	0.05	0.06
100	5	2	0.01	0.02	0.03	0.06	0.07
100	5	2	0.02	0.02	0.03	0.06	0.07
100	5	2	0.03	0.03	0.03	0.06	0.07
100	5	2	0.05	0.03	0.03	0.06	0.07
100	5	2	0.1	0.03	0.04	0.05	0.08
100	5	2	0.2	0.04	0.04	0.05	0.08
200	5	2	0.01	0.02	0.03	0.06	0.06
200	5	2	0.02	0.03	0.03	0.06	0.06
200	5	2	0.03	0.03	0.03	0.05	0.06
200	5	2	0.05	0.03	0.03	0.05	0.06
200	5	2	0.1	0.04	0.04	0.05	0.06
200	5	2	0.2	0.04	0.05	0.05	0.06
500	5	2	0.01	0.03	0.03	0.06	0.05
500	5	2	0.02	0.03	0.03	0.05	0.05
500	5	2	0.03	0.04	0.04	0.05	0.05
500	5	2	0.05	0.04	0.04	0.05	0.05
500	5	2	0.1	0.04	0.04	0.05	0.05
500	5	2	0.2	0.05	0.04	0.05	0.05
100	5	5	0.01	0.02	0.02	0.06	0.08
100	5	5	0.02	0.02	0.02	0.06	0.08
100	5	5	0.03	0.02	0.02	0.06	0.08
100	5	5	0.05	0.02	0.03	0.05	0.08
100	5	5	0.1	0.03	0.03	0.05	0.09
100	5	5	0.2	0.03	0.04	0.05	0.09
200	5	5	0.01	0.02	0.02	0.06	0.06
200	5	5	0.02	0.02	0.03	0.06	0.06
200	5	5	0.03	0.03	0.03	0.06	0.06
200	5	5	0.05	0.03	0.03	0.06	0.06
200	5	5	0.1	0.04	0.04	0.05	0.06
200	5	5	0.2	0.04	0.04	0.05	0.06
500	5	5	0.01	0.03	0.03	0.06	0.06
500	5	5	0.02	0.04	0.04	0.06	0.06
500	5	5	0.03	0.04	0.04	0.06	0.06
500	5	5	0.05	0.04	0.04	0.05	0.06
500	5	5	0.1	0.05	0.05	0.05	0.06
500	5	5	0.2	0.05	0.05	0.05	0.06

Table 5: Size, ρ=.50

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	2	1	0.01	0.04	0.03	0.07	0.07
100	2	1	0.02	0.04	0.04	0.07	0.07
100	2	1	0.03	0.04	0.04	0.06	0.07
100	2	1	0.05	0.05	0.04	0.05	0.07
100	2	1	0.1	0.06	0.04	0.05	0.07
100	2	1	0.2	0.06	0.04	0.05	0.06
200	2	1	0.01	0.04	0.04	0.06	0.06
200	2	1	0.02	0.05	0.04	0.06	0.06
200	2	1	0.03	0.05	0.05	0.05	0.06
200	2	1	0.05	0.05	0.05	0.05	0.06
200	2	1	0.1	0.05	0.04	0.05	0.06
200	2	1	0.2	0.06	0.04	0.05	0.06
500	2	1	0.01	0.05	0.05	0.06	0.06
500	2	1	0.02	0.06	0.05	0.06	0.06
500	2	1	0.03	0.06	0.05	0.06	0.06
500	2	1	0.05	0.06	0.05	0.06	0.06
500	2	1	0.1	0.06	0.05	0.06	0.06
500	2	1	0.2	0.06	0.05	0.05	0.06
100	2	2	0.01	0.03	0.03	0.06	0.06
100	2	2	0.02	0.04	0.04	0.06	0.06
100	2	2	0.03	0.05	0.04	0.06	0.06
100	2	2	0.05	0.05	0.04	0.05	0.06
100	2	2	0.1	0.05	0.04	0.05	0.06
100	2	2	0.2	0.05	0.04	0.05	0.06
200	2	2	0.01	0.04	0.04	0.06	0.05
200	2	2	0.02	0.05	0.04	0.06	0.05
200	2	2	0.03	0.05	0.05	0.05	0.05
200	2	2	0.05	0.05	0.05	0.05	0.05
200	2	2	0.1	0.05	0.04	0.05	0.06
200	2	2	0.2	0.06	0.04	0.05	0.06
500	2	2	0.01	0.05	0.05	0.06	0.06
500	2	2	0.02	0.06	0.05	0.06	0.06
500	2	2	0.03	0.05	0.05	0.05	0.06
500	2	2	0.05	0.05	0.05	0.05	0.06
500	2	2	0.1	0.05	0.05	0.05	0.06
500	2	2	0.2	0.05	0.05	0.06	0.06
100	2	5	0.01	0.02	0.02	0.07	0.07
100	2	5	0.02	0.03	0.03	0.06	0.07
100	2	5	0.03	0.03	0.03	0.06	0.07
100	2	5	0.05	0.04	0.03	0.05	0.07
100	2	5	0.1	0.04	0.03	0.05	0.07
100	2	5	0.2	0.04	0.03	0.06	0.07
200	2	5	0.01	0.03	0.03	0.06	0.07
200	2	5	0.02	0.04	0.03	0.06	0.06
200	2	5	0.03	0.04	0.04	0.06	0.06
200	2	5	0.05	0.04	0.04	0.06	0.06
200	2	5	0.1	0.04	0.04	0.06	0.06
200	2	5	0.2	0.04	0.04	0.06	0.07
500	2	5	0.01	0.04	0.04	0.05	0.05
500	2	5	0.02	0.05	0.04	0.05	0.05
500	2	5	0.03	0.04	0.04	0.05	0.05
500	2	5	0.05	0.04	0.04	0.05	0.05
500	2	5	0.1	0.04	0.04	0.05	0.05
500	2	5	0.2	0.05	0.04	0.05	0.05

Table 5 - Continued: Size, ρ=.50

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	5	1	0.01	0.11	0.10	0.12	0.08
100	5	1	0.02	0.12	0.10	0.11	0.08
100	5	1	0.03	0.12	0.10	0.10	0.08
100	5	1	0.05	0.11	0.09	0.08	0.09
100	5	1	0.1	0.10	0.07	0.07	0.08
100	5	1	0.2	0.08	0.06	0.06	0.08
200	5	1	0.01	0.13	0.12	0.13	0.08
200	5	1	0.02	0.12	0.12	0.10	0.08
200	5	1	0.03	0.12	0.11	0.09	0.08
200	5	1	0.05	0.10	0.09	0.08	0.08
200	5	1	0.1	0.09	0.07	0.06	0.08
200	5	1	0.2	0.07	0.05	0.05	0.08
500	5	1	0.01	0.12	0.11	0.09	0.06
500	5	1	0.02	0.10	0.10	0.08	0.06
500	5	1	0.03	0.09	0.09	0.07	0.06
500	5	1	0.05	0.08	0.07	0.06	0.06
500	5	1	0.1	0.06	0.06	0.05	0.06
500	5	1	0.2	0.06	0.05	0.05	0.06
100	5	2	0.01	0.11	0.10	0.12	0.09
100	5	2	0.02	0.11	0.10	0.11	0.09
100	5	2	0.03	0.10	0.09	0.10	0.09
100	5	2	0.05	0.10	0.09	0.09	0.09
100	5	2	0.1	0.09	0.07	0.07	0.09
100	5	2	0.2	0.07	0.05	0.05	0.09
200	5	2	0.01	0.11	0.10	0.11	0.07
200	5	2	0.02	0.11	0.10	0.09	0.07
200	5	2	0.03	0.10	0.09	0.08	0.07
200	5	2	0.05	0.09	0.08	0.07	0.07
200	5	2	0.1	0.07	0.06	0.06	0.07
200	5	2	0.2	0.06	0.05	0.05	0.07
500	5	2	0.01	0.12	0.11	0.10	0.05
500	5	2	0.02	0.10	0.10	0.07	0.05
500	5	2	0.03	0.09	0.08	0.06	0.05
500	5	2	0.05	0.07	0.07	0.06	0.06
500	5	2	0.1	0.06	0.06	0.05	0.06
500	5	2	0.2	0.05	0.05	0.05	0.05
100	5	5	0.01	0.07	0.07	0.10	0.10
100	5	5	0.02	0.07	0.07	0.09	0.09
100	5	5	0.03	0.07	0.07	0.08	0.09
100	5	5	0.05	0.07	0.06	0.07	0.10
100	5	5	0.1	0.05	0.05	0.05	0.10
100	5	5	0.2	0.05	0.04	0.05	0.10
200	5	5	0.01	0.09	0.09	0.09	0.07
200	5	5	0.02	0.09	0.08	0.08	0.07
200	5	5	0.03	0.08	0.08	0.07	0.07
200	5	5	0.05	0.08	0.07	0.06	0.07
200	5	5	0.1	0.06	0.05	0.05	0.07
200	5	5	0.2	0.05	0.04	0.05	0.07
500	5	5	0.01	0.11	0.11	0.09	0.07
500	5	5	0.02	0.09	0.09	0.07	0.07
500	5	5	0.03	0.08	0.08	0.07	0.07
500	5	5	0.05	0.07	0.06	0.06	0.07
500	5	5	0.1	0.06	0.05	0.05	0.07
500	5	5	0.2	0.05	0.05	0.05	0.07

Table 6: Size, ρ=.75

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	2	1	0.01	0.15	0.13	0.10	0.08
100	2	1	0.02	0.15	0.12	0.09	0.08
100	2	1	0.03	0.14	0.11	0.08	0.08
100	2	1	0.05	0.13	0.10	0.06	0.08
100	2	1	0.1	0.11	0.07	0.05	0.07
100	2	1	0.2	0.09	0.05	0.05	0.07
200	2	1	0.01	0.15	0.13	0.08	0.07
200	2	1	0.02	0.14	0.12	0.07	0.07
200	2	1	0.03	0.12	0.10	0.06	0.07
200	2	1	0.05	0.11	0.08	0.05	0.07
200	2	1	0.1	0.09	0.06	0.05	0.06
200	2	1	0.2	0.08	0.05	0.05	0.06
500	2	1	0.01	0.12	0.11	0.07	0.06
500	2	1	0.02	0.10	0.09	0.06	0.06
500	2	1	0.03	0.09	0.08	0.06	0.06
500	2	1	0.05	0.08	0.07	0.06	0.06
500	2	1	0.1	0.07	0.06	0.06	0.06
500	2	1	0.2	0.07	0.06	0.05	0.06
100	2	2	0.01	0.14	0.11	0.09	0.08
100	2	2	0.02	0.14	0.11	0.07	0.07
100	2	2	0.03	0.13	0.10	0.07	0.07
100	2	2	0.05	0.11	0.08	0.06	0.07
100	2	2	0.1	0.10	0.06	0.06	0.07
100	2	2	0.2	0.08	0.04	0.05	0.07
200	2	2	0.01	0.14	0.12	0.08	0.06
200	2	2	0.02	0.13	0.11	0.06	0.06
200	2	2	0.03	0.12	0.10	0.06	0.06
200	2	2	0.05	0.10	0.08	0.05	0.06
200	2	2	0.1	0.08	0.05	0.05	0.06
200	2	2	0.2	0.07	0.04	0.05	0.06
500	2	2	0.01	0.11	0.11	0.07	0.06
500	2	2	0.02	0.09	0.09	0.06	0.06
500	2	2	0.03	0.08	0.07	0.06	0.06
500	2	2	0.05	0.07	0.06	0.05	0.06
500	2	2	0.1	0.06	0.05	0.05	0.06
500	2	2	0.2	0.06	0.05	0.06	0.06
100	2	5	0.01	0.09	0.08	0.08	0.08
100	2	5	0.02	0.09	0.07	0.07	0.08
100	2	5	0.03	0.08	0.06	0.06	0.08
100	2	5	0.05	0.07	0.05	0.05	0.08
100	2	5	0.1	0.05	0.04	0.06	0.08
100	2	5	0.2	0.04	0.03	0.06	0.08
200	2	5	0.01	0.11	0.10	0.07	0.07
200	2	5	0.02	0.10	0.09	0.06	0.07
200	2	5	0.03	0.09	0.07	0.06	0.07
200	2	5	0.05	0.08	0.06	0.06	0.07
200	2	5	0.1	0.06	0.04	0.06	0.07
200	2	5	0.2	0.05	0.04	0.06	0.07
500	2	5	0.01	0.11	0.10	0.05	0.06
500	2	5	0.02	0.08	0.08	0.05	0.06
500	2	5	0.03	0.07	0.06	0.05	0.06
500	2	5	0.05	0.06	0.05	0.05	0.06
500	2	5	0.1	0.05	0.04	0.05	0.05
500	2	5	0.2	0.05	0.04	0.05	0.05

Table 6 - Continued: Size, ρ=.75

n	# weak IV	# strong IV	R^2	H1	H2	H3	H4
100	5	1	0.01	0.37	0.32	0.23	0.11
100	5	1	0.02	0.34	0.29	0.20	0.11
100	5	1	0.03	0.31	0.27	0.18	0.11
100	5	1	0.05	0.27	0.22	0.14	0.11
100	5	1	0.1	0.20	0.15	0.10	0.11
100	5	1	0.2	0.14	0.09	0.07	0.10
200	5	1	0.01	0.38	0.35	0.23	0.09
200	5	1	0.02	0.33	0.30	0.19	0.09
200	5	1	0.03	0.29	0.25	0.15	0.09
200	5	1	0.05	0.24	0.19	0.12	0.09
200	5	1	0.1	0.16	0.12	0.08	0.09
200	5	1	0.2	0.11	0.07	0.06	0.09
500	5	1	0.01	0.30	0.28	0.16	0.07
500	5	1	0.02	0.22	0.21	0.12	0.07
500	5	1	0.03	0.18	0.16	0.10	0.07
500	5	1	0.05	0.14	0.12	0.07	0.07
500	5	1	0.1	0.10	0.08	0.06	0.07
500	5	1	0.2	0.07	0.06	0.05	0.07
100	5	2	0.01	0.35	0.31	0.23	0.12
100	5	2	0.02	0.33	0.28	0.20	0.12
100	5	2	0.03	0.30	0.25	0.17	0.11
100	5	2	0.05	0.25	0.20	0.14	0.11
100	5	2	0.1	0.18	0.13	0.09	0.11
100	5	2	0.2	0.12	0.07	0.06	0.10
200	5	2	0.01	0.35	0.32	0.20	0.09
200	5	2	0.02	0.30	0.27	0.16	0.08
200	5	2	0.03	0.26	0.22	0.13	0.08
200	5	2	0.05	0.21	0.17	0.10	0.08
200	5	2	0.1	0.14	0.10	0.07	0.08
200	5	2	0.2	0.09	0.06	0.06	0.08
500	5	2	0.01	0.30	0.28	0.16	0.06
500	5	2	0.02	0.22	0.20	0.11	0.06
500	5	2	0.03	0.18	0.16	0.09	0.06
500	5	2	0.05	0.13	0.11	0.07	0.06
500	5	2	0.1	0.09	0.07	0.06	0.06
500	5	2	0.2	0.07	0.06	0.05	0.06
100	5	5	0.01	0.26	0.23	0.18	0.12
100	5	5	0.02	0.23	0.20	0.15	0.12
100	5	5	0.03	0.21	0.17	0.13	0.12
100	5	5	0.05	0.17	0.13	0.09	0.12
100	5	5	0.1	0.11	0.07	0.06	0.12
100	5	5	0.2	0.06	0.04	0.04	0.12
200	5	5	0.01	0.30	0.27	0.18	0.09
200	5	5	0.02	0.25	0.22	0.13	0.09
200	5	5	0.03	0.20	0.18	0.11	0.09
200	5	5	0.05	0.15	0.13	0.08	0.08
200	5	5	0.1	0.09	0.07	0.06	0.08
200	5	5	0.2	0.06	0.04	0.05	0.08
500	5	5	0.01	0.27	0.26	0.16	0.07
500	5	5	0.02	0.20	0.19	0.11	0.07
500	5	5	0.03	0.16	0.14	0.09	0.07
500	5	5	0.05	0.11	0.10	0.07	0.07
500	5	5	0.1	0.08	0.06	0.06	0.07
500	5	5	0.2	0.06	0.05	0.05	0.07

Table 7: Power, # weak IV = 1, γ_s=1

n	# strong IV	R^2	ρ	H3	H4
100	1	0.01	0.25	0.20	0.21
100	1	0.02	0.25	0.33	0.34
100	1	0.03	0.25	0.45	0.46
100	1	0.05	0.25	0.64	0.66
100	1	0.1	0.25	0.91	0.91
100	1	0.2	0.25	1.00	1.00
200	1	0.01	0.25	0.30	0.31
200	1	0.02	0.25	0.53	0.53
200	1	0.03	0.25	0.70	0.71
200	1	0.05	0.25	0.89	0.90
200	1	0.1	0.25	1.00	1.00
200	1	0.2	0.25	1.00	1.00
500	1	0.01	0.25	0.62	0.63
500	1	0.02	0.25	0.90	0.90
500	1	0.03	0.25	0.97	0.97
500	1	0.05	0.25	1.00	1.00
500	1	0.1	0.25	1.00	1.00
500	1	0.2	0.25	1.00	1.00
100	1	0.01	0.5	0.24	0.25
100	1	0.02	0.5	0.41	0.42
100	1	0.03	0.5	0.55	0.57
100	1	0.05	0.5	0.76	0.77
100	1	0.1	0.5	0.96	0.97
100	1	0.2	0.5	1.00	1.00
200	1	0.01	0.5	0.38	0.39
200	1	0.02	0.5	0.65	0.65
200	1	0.03	0.5	0.82	0.82
200	1	0.05	0.5	0.96	0.96
200	1	0.1	0.5	1.00	1.00
200	1	0.2	0.5	1.00	1.00
500	1	0.01	0.5	0.74	0.75
500	1	0.02	0.5	0.96	0.96
500	1	0.03	0.5	0.99	0.99
500	1	0.05	0.5	1.00	1.00
500	1	0.1	0.5	1.00	1.00
500	1	0.2	0.5	1.00	1.00
100	1	0.01	0.75	0.33	0.34
100	1	0.02	0.75	0.55	0.56
100	1	0.03	0.75	0.72	0.73
100	1	0.05	0.75	0.90	0.90
100	1	0.1	0.75	0.99	0.99
100	1	0.2	0.75	1.00	1.00
200	1	0.01	0.75	0.54	0.54
200	1	0.02	0.75	0.81	0.82
200	1	0.03	0.75	0.94	0.94
200	1	0.05	0.75	0.99	1.00
200	1	0.1	0.75	1.00	1.00
200	1	0.2	0.75	1.00	1.00
500	1	0.01	0.75	0.89	0.89
500	1	0.02	0.75	0.99	0.99
500	1	0.03	0.75	1.00	1.00
500	1	0.05	0.75	1.00	1.00
500	1	0.1	0.75	1.00	1.00
500	1	0.2	0.75	1.00	1.00

Table 7 - Continued: Power, # weak IV = 1, γ_s=1

n	# strong IV	R^2	ρ	H3	H4
100	2	0.01	0.25	0.15	0.21
100	2	0.02	0.25	0.23	0.30
100	2	0.03	0.25	0.31	0.40
100	2	0.05	0.25	0.45	0.55
100	2	0.1	0.25	0.71	0.80
100	2	0.2	0.25	0.90	0.95
200	2	0.01	0.25	0.22	0.29
200	2	0.02	0.25	0.38	0.46
200	2	0.03	0.25	0.52	0.61
200	2	0.05	0.25	0.74	0.80
200	2	0.1	0.25	0.94	0.97
200	2	0.2	0.25	0.99	1.00
500	2	0.01	0.25	0.47	0.55
500	2	0.02	0.25	0.74	0.80
500	2	0.03	0.25	0.90	0.93
500	2	0.05	0.25	0.98	0.99
500	2	0.1	0.25	1.00	1.00
500	2	0.2	0.25	1.00	1.00
100	2	0.01	0.5	0.17	0.26
100	2	0.02	0.5	0.27	0.38
100	2	0.03	0.5	0.37	0.50
100	2	0.05	0.5	0.55	0.67
100	2	0.1	0.5	0.80	0.89
100	2	0.2	0.5	0.94	0.97
200	2	0.01	0.5	0.27	0.37
200	2	0.02	0.5	0.47	0.59
200	2	0.03	0.5	0.64	0.75
200	2	0.05	0.5	0.84	0.91
200	2	0.1	0.5	0.98	0.99
200	2	0.2	0.5	1.00	1.00
500	2	0.01	0.5	0.58	0.67
500	2	0.02	0.5	0.86	0.92
500	2	0.03	0.5	0.96	0.98
500	2	0.05	0.5	1.00	1.00
500	2	0.1	0.5	1.00	1.00
500	2	0.2	0.5	1.00	1.00
100	2	0.01	0.75	0.21	0.37
100	2	0.02	0.75	0.37	0.55
100	2	0.03	0.75	0.51	0.68
100	2	0.05	0.75	0.71	0.84
100	2	0.1	0.75	0.90	0.96
100	2	0.2	0.75	0.96	0.99
200	2	0.01	0.75	0.38	0.55
200	2	0.02	0.75	0.65	0.80
200	2	0.03	0.75	0.81	0.91
200	2	0.05	0.75	0.95	0.98
200	2	0.1	0.75	0.99	1.00
200	2	0.2	0.75	1.00	1.00
500	2	0.01	0.75	0.75	0.87
500	2	0.02	0.75	0.96	0.98
500	2	0.03	0.75	0.99	1.00
500	2	0.05	0.75	1.00	1.00
500	2	0.1	0.75	1.00	1.00
500	2	0.2	0.75	1.00	1.00

Table 7 - Continued: Power, # weak IV = 1, γ_s=1

n	# strong IV	R^2	ρ	H3	H4
100	5	0.01	0.25	0.10	0.22
100	5	0.02	0.25	0.14	0.27
100	5	0.03	0.25	0.17	0.32
100	5	0.05	0.25	0.24	0.40
100	5	0.1	0.25	0.37	0.56
100	5	0.2	0.25	0.55	0.71
200	5	0.01	0.25	0.13	0.25
200	5	0.02	0.25	0.21	0.36
200	5	0.03	0.25	0.29	0.45
200	5	0.05	0.25	0.43	0.59
200	5	0.1	0.25	0.64	0.78
200	5	0.2	0.25	0.81	0.90
500	5	0.01	0.25	0.27	0.42
500	5	0.02	0.25	0.46	0.62
500	5	0.03	0.25	0.61	0.76
500	5	0.05	0.25	0.79	0.89
500	5	0.1	0.25	0.95	0.98
500	5	0.2	0.25	0.99	1.00
100	5	0.01	0.5	0.11	0.26
100	5	0.02	0.5	0.16	0.33
100	5	0.03	0.5	0.21	0.39
100	5	0.05	0.5	0.29	0.50
100	5	0.1	0.5	0.45	0.65
100	5	0.2	0.5	0.62	0.78
200	5	0.01	0.5	0.16	0.32
200	5	0.02	0.5	0.26	0.46
200	5	0.03	0.5	0.35	0.56
200	5	0.05	0.5	0.50	0.69
200	5	0.1	0.5	0.71	0.86
200	5	0.2	0.5	0.86	0.93
500	5	0.01	0.5	0.32	0.53
500	5	0.02	0.5	0.54	0.74
500	5	0.03	0.5	0.70	0.85
500	5	0.05	0.5	0.86	0.95
500	5	0.1	0.5	0.97	0.99
500	5	0.2	0.5	0.99	1.00
100	5	0.01	0.75	0.13	0.36
100	5	0.02	0.75	0.20	0.45
100	5	0.03	0.75	0.27	0.54
100	5	0.05	0.75	0.37	0.63
100	5	0.1	0.75	0.55	0.77
100	5	0.2	0.75	0.70	0.84
200	5	0.01	0.75	0.20	0.46
200	5	0.02	0.75	0.33	0.61
200	5	0.03	0.75	0.45	0.71
200	5	0.05	0.75	0.61	0.82
200	5	0.1	0.75	0.80	0.91
200	5	0.2	0.75	0.89	0.95
500	5	0.01	0.75	0.41	0.70
500	5	0.02	0.75	0.67	0.88
500	5	0.03	0.75	0.81	0.94
500	5	0.05	0.75	0.93	0.98
500	5	0.1	0.75	0.98	0.99
500	5	0.2	0.75	1.00	1.00

Table 8: Power, # weak IV = 1, γ_s=2

n	# strong IV	R^2	ρ	H3	H4
100	1	0.01	0.25	0.22	0.23
100	1	0.02	0.25	0.36	0.37
100	1	0.03	0.25	0.49	0.50
100	1	0.05	0.25	0.69	0.70
100	1	0.1	0.25	0.93	0.93
100	1	0.2	0.25	1.00	1.00
200	1	0.01	0.25	0.33	0.33
200	1	0.02	0.25	0.56	0.57
200	1	0.03	0.25	0.74	0.74
200	1	0.05	0.25	0.91	0.92
200	1	0.1	0.25	1.00	1.00
200	1	0.2	0.25	1.00	1.00
500	1	0.01	0.25	0.65	0.66
500	1	0.02	0.25	0.92	0.92
500	1	0.03	0.25	0.98	0.98
500	1	0.05	0.25	1.00	1.00
500	1	0.1	0.25	1.00	1.00
500	1	0.2	0.25	1.00	1.00
100	1	0.01	0.5	0.24	0.25
100	1	0.02	0.5	0.40	0.41
100	1	0.03	0.5	0.54	0.55
100	1	0.05	0.5	0.75	0.76
100	1	0.1	0.5	0.96	0.96
100	1	0.2	0.5	1.00	1.00
200	1	0.01	0.5	0.37	0.38
200	1	0.02	0.5	0.63	0.63
200	1	0.03	0.5	0.79	0.80
200	1	0.05	0.5	0.95	0.95
200	1	0.1	0.5	1.00	1.00
200	1	0.2	0.5	1.00	1.00
500	1	0.01	0.5	0.71	0.72
500	1	0.02	0.5	0.95	0.95
500	1	0.03	0.5	0.99	0.99
500	1	0.05	0.5	1.00	1.00
500	1	0.1	0.5	1.00	1.00
500	1	0.2	0.5	1.00	1.00
100	1	0.01	0.75	0.27	0.28
100	1	0.02	0.75	0.46	0.47
100	1	0.03	0.75	0.61	0.62
100	1	0.05	0.75	0.80	0.81
100	1	0.1	0.75	0.98	0.98
100	1	0.2	0.75	1.00	1.00
200	1	0.01	0.75	0.43	0.44
200	1	0.02	0.75	0.70	0.70
200	1	0.03	0.75	0.86	0.86
200	1	0.05	0.75	0.97	0.97
200	1	0.1	0.75	1.00	1.00
200	1	0.2	0.75	1.00	1.00
500	1	0.01	0.75	0.78	0.79
500	1	0.02	0.75	0.97	0.97
500	1	0.03	0.75	1.00	1.00
500	1	0.05	0.75	1.00	1.00
500	1	0.1	0.75	1.00	1.00
500	1	0.2	0.75	1.00	1.00

Table 8 - Continued: Power, # weak IV = 1, γ_s=2

n	# strong IV	R^2	ρ	H3	H4
100	2	0.01	0.25	0.16	0.25
100	2	0.02	0.25	0.26	0.37
100	2	0.03	0.25	0.36	0.48
100	2	0.05	0.25	0.53	0.66
100	2	0.1	0.25	0.80	0.89
100	2	0.2	0.25	0.96	0.99
200	2	0.01	0.25	0.26	0.34
200	2	0.02	0.25	0.44	0.55
200	2	0.03	0.25	0.60	0.71
200	2	0.05	0.25	0.82	0.89
200	2	0.1	0.25	0.98	0.99
200	2	0.2	0.25	1.00	1.00
500	2	0.01	0.25	0.53	0.62
500	2	0.02	0.25	0.82	0.88
500	2	0.03	0.25	0.94	0.97
500	2	0.05	0.25	0.99	1.00
500	2	0.1	0.25	1.00	1.00
500	2	0.2	0.25	1.00	1.00
100	2	0.01	0.5	0.18	0.28
100	2	0.02	0.5	0.28	0.42
100	2	0.03	0.5	0.40	0.55
100	2	0.05	0.5	0.59	0.73
100	2	0.1	0.5	0.85	0.93
100	2	0.2	0.5	0.97	0.99
200	2	0.01	0.5	0.29	0.40
200	2	0.02	0.5	0.50	0.63
200	2	0.03	0.5	0.68	0.79
200	2	0.05	0.5	0.88	0.93
200	2	0.1	0.5	0.99	1.00
200	2	0.2	0.5	1.00	1.00
500	2	0.01	0.5	0.59	0.70
500	2	0.02	0.5	0.88	0.93
500	2	0.03	0.5	0.97	0.98
500	2	0.05	0.5	1.00	1.00
500	2	0.1	0.5	1.00	1.00
500	2	0.2	0.5	1.00	1.00
100	2	0.01	0.75	0.20	0.34
100	2	0.02	0.75	0.33	0.50
100	2	0.03	0.75	0.46	0.63
100	2	0.05	0.75	0.66	0.81
100	2	0.1	0.75	0.90	0.97
100	2	0.2	0.75	0.98	1.00
200	2	0.01	0.75	0.34	0.49
200	2	0.02	0.75	0.59	0.73
200	2	0.03	0.75	0.76	0.86
200	2	0.05	0.75	0.93	0.97
200	2	0.1	0.75	1.00	1.00
200	2	0.2	0.75	1.00	1.00
500	2	0.01	0.75	0.69	0.79
500	2	0.02	0.75	0.93	0.97
500	2	0.03	0.75	0.99	0.99
500	2	0.05	0.75	1.00	1.00
500	2	0.1	0.75	1.00	1.00
500	2	0.2	0.75	1.00	1.00

Table 8 - Continued: Power, # weak IV = 1, γ_s=2

n	# strong IV	R^2	ρ	H3	H4
100	5	0.01	0.25	0.11	0.32
100	5	0.02	0.25	0.16	0.40
100	5	0.03	0.25	0.22	0.46
100	5	0.05	0.25	0.30	0.57
100	5	0.1	0.25	0.47	0.74
100	5	0.2	0.25	0.67	0.87
200	5	0.01	0.25	0.16	0.37
200	5	0.02	0.25	0.26	0.51
200	5	0.03	0.25	0.36	0.62
200	5	0.05	0.25	0.53	0.77
200	5	0.1	0.25	0.76	0.92
200	5	0.2	0.25	0.90	0.98
500	5	0.01	0.25	0.33	0.57
500	5	0.02	0.25	0.57	0.79
500	5	0.03	0.25	0.73	0.89
500	5	0.05	0.25	0.89	0.97
500	5	0.1	0.25	0.98	1.00
500	5	0.2	0.25	1.00	1.00
100	5	0.01	0.5	0.12	0.38
100	5	0.02	0.5	0.18	0.46
100	5	0.03	0.5	0.23	0.54
100	5	0.05	0.5	0.34	0.64
100	5	0.1	0.5	0.53	0.81
100	5	0.2	0.5	0.71	0.90
200	5	0.01	0.5	0.18	0.44
200	5	0.02	0.5	0.30	0.59
200	5	0.03	0.5	0.41	0.70
200	5	0.05	0.5	0.58	0.84
200	5	0.1	0.5	0.80	0.95
200	5	0.2	0.5	0.91	0.98
500	5	0.01	0.5	0.38	0.65
500	5	0.02	0.5	0.63	0.85
500	5	0.03	0.5	0.79	0.94
500	5	0.05	0.5	0.92	0.99
500	5	0.1	0.5	0.99	1.00
500	5	0.2	0.5	1.00	1.00
100	5	0.01	0.75	0.13	0.46
100	5	0.02	0.75	0.19	0.56
100	5	0.03	0.75	0.26	0.63
100	5	0.05	0.75	0.38	0.75
100	5	0.1	0.75	0.58	0.87
100	5	0.2	0.75	0.75	0.93
200	5	0.01	0.75	0.20	0.55
200	5	0.02	0.75	0.34	0.70
200	5	0.03	0.75	0.47	0.81
200	5	0.05	0.75	0.65	0.91
200	5	0.1	0.75	0.85	0.97
200	5	0.2	0.75	0.93	0.99
500	5	0.01	0.75	0.43	0.76
500	5	0.02	0.75	0.69	0.92
500	5	0.03	0.75	0.84	0.97
500	5	0.05	0.75	0.95	1.00
500	5	0.1	0.75	0.99	1.00
500	5	0.2	0.75	1.00	1.00