

**Note on Omitted Proofs for “Incidental
Trends and the Power of Panel Unit
Root Tests”**

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Lemma 5 Suppose that Assumption 1 is satisfied. Then, as $n, T \rightarrow \infty$ with $\frac{n}{T} \rightarrow 0$, the following hold.

- (a) $\sum_{i=1}^n (\tilde{\sigma}_{iT}^2 - \sigma_i^2)^2 = o_p(1)$.
- (b) $\sup_{1 \leq i \leq n} |\tilde{\sigma}_{iT}^2 - \sigma_i^2| = o_p(1)$.
- (c) With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \tilde{\sigma}_{iT}^2 \geq M$.

Proof

Let $X_{iT} = \frac{1}{\sqrt{T}} \sum_{t=1}^T (u_{it}^2 - \sigma_i^2)$. Then, $EX_{iT}^2 = \text{Var}(u_{it}^2) \leq E(u_{it}^4) \leq \bar{M}$.

Part (a): The required result follows by the Markov inequality and

$$E \left[\sum_{i=1}^n (\tilde{\sigma}_{iT}^2 - \sigma_i^2)^2 \right] = \frac{n}{T} \frac{1}{n} \sum_{i=1}^n EX_{iT}^2 \leq \bar{M} \frac{n}{T} \rightarrow 0. \blacksquare$$

Part (b): The required result follows if we show that $\sup_{1 \leq i \leq n} (\tilde{\sigma}_{iT}^2 - \sigma_i^2)^2 = o_p(1)$, which holds because $\sup_{1 \leq i \leq n} (\tilde{\sigma}_{iT}^2 - \sigma_i^2)^2 \leq \sum_{i=1}^n (\tilde{\sigma}_{iT}^2 - \sigma_i^2)^2 = o_p(1)$ by Part (a). \blacksquare

Part (c): The required result follows since

$$\inf_{1 \leq i \leq n} \tilde{\sigma}_{iT}^2 \geq \inf_{1 \leq i \leq n} \sigma_i^2 - \sup_{1 \leq i \leq n} |\tilde{\sigma}_{iT}^2 - \sigma_i^2|$$

and by Part (b) and Assumption 1. \blacksquare

Suppose that c_i is a sequence of *iid* random variables with the same support as θ_i and that c_i is independent of u_{it} for all i and t .

Lemma 6 Suppose that Assumptions 1–3, 6, and 7 hold. Then, the following hold as $(n, T \rightarrow \infty)$ with $\frac{n}{T} \rightarrow 0$.

- (a) $\frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left\{ (y_{it} - y_{i0}) - \frac{t}{T} (y_{iT} - y_{i0}) \right\}^2 - \omega_{1T} \right] \Rightarrow N \left(-\frac{E(c_i^2 \theta_i^2)}{90}, \frac{E(c_i^4)}{45} \right)$
- (b) $\frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it} y_{is} h_T(t, s) - \omega_{2T} \right] \Rightarrow N \left(-\frac{E(\theta_i^2)}{420}, \frac{11}{6300} \right)$.

Proof:

Part (a): For notational simplicity let $\bar{Y}_{it,T} = (y_{it} - y_{i0}) - \frac{t}{T} (y_{iT} - y_{i0})$ and $\bar{Y}_{it,T}(0) = (y_{it}(0) - y_{i0}(0)) - \frac{t}{T} (y_{iT}(0) - y_{i0}(0))$. Using this notation, we de-

compose

$$\begin{aligned}
& \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left\{ (y_{it} - y_{i0}) - \frac{t}{T} (y_{iT} - y_{i0}) \right\}^2 - \omega_{1T} \right] \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) + \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T (\bar{Y}_{it,T} - \bar{Y}_{it,T}(0))^2 \right) \\
& + \frac{2}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}(0) (\bar{Y}_{it,T} - \bar{Y}_{it,T}(0)) \right) \\
= & I_a + II_a + III_a, \text{ say.}
\end{aligned}$$

By a direct calculation

$$E \left[c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right] = O \left(\frac{1}{T} \right).$$

Also, it is possible to show that

$$\frac{1}{n} \sum_{i=1}^n \text{Var} \left[c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right] \rightarrow \frac{1}{45} E(c_i^4),$$

and

$$\sup_{i,T} E \left[c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}^2(0) - \omega_{1T} \right) \right]^4 < M.$$

Then, since $\frac{n}{T} \rightarrow 0$, by the double-indexed CLT in Phillips and Moon (1999a), we have

$$I_a \Rightarrow N \left(0, \frac{1}{45} E(c_i^4) \right). \quad (1)$$

For term II_a , by definition we have

$$\begin{aligned}
& II_a \\
= & \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T (y_{it} - y_{it}(0))^2 \right) \\
& - \frac{2}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left(\frac{t}{T} \right) (y_{it} - y_{it}(0)) (y_{iT} - y_{iT}(0)) \right) \\
& + \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left(\frac{t}{T} \right)^2 (y_{iT} - y_{iT}(0))^2 \right) \\
= & II_{a1} + II_{a2} + II_{a3}, \text{ say.}
\end{aligned}$$

By definition again

$$\begin{aligned}
y_{it} - y_{it}(0) &= \sum_{p=0}^{t-1} (\rho_i^{t-p} - 1) u_{ip} = \sum_{p=0}^{t-1} \left[\sum_{l=1}^{t-p} \binom{t-p}{l} \left(\frac{-\theta_i}{n^\kappa T} \right)^l \right] u_{ip} \text{ for } t \geq 1 \\
&= 0 \text{ for } t = 0.
\end{aligned} \quad (2)$$

Recall that $\kappa = \frac{1}{4}$. By (2) and by the WLLN, we have

$$\begin{aligned} II_{a1} &\sim \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \theta_i^2 \frac{1}{T\sigma_i^2} \sum_{t=1}^T \left(\frac{1}{\sqrt{T}} \sum_{p=0}^{t-1} \left(\frac{t-p}{T} \right) u_{ip} \right)^2 \\ &\rightarrow_p E(c_i^2 \theta_i^2) \int_0^1 \int_0^r (r-s)^2 ds dr = \frac{1}{12} E(c_i^2 \theta_i^2), \end{aligned}$$

$$\begin{aligned} II_{a2} &\sim -\frac{2}{\sqrt{n}} \sum_{i=1}^n c_i^2 \theta_i^2 \left(\frac{1}{T\sigma_i^2} \sum_{t=1}^T \left(\frac{t-1}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^{t-1} \left(\frac{t-p}{T} \right) u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T} \right) u_{iq} \right) \right) \\ &\rightarrow_p -2E(c_i^2 \theta_i^2) \int_0^1 r \int_0^r (r-s)(1-s) ds dr = -\frac{11}{60} E(c_i^2 \theta_i^2), \end{aligned}$$

and

$$\begin{aligned} II_{a3} &\sim \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \theta_i^2 \frac{1}{T\sigma_i^2} \sum_{t=1}^T \left(\frac{t}{T} \right)^2 \left(\frac{1}{\sqrt{T}} \sum_{p=0}^{T-1} \left(\frac{T-p}{T} \right) u_{ip} \right)^2 \\ &\rightarrow_p E(c_i^2 \theta_i^2) \int_0^1 r^2 dr \int_0^1 (1-r)^2 dr = \frac{1}{9} E(c_i^2 \theta_i^2). \end{aligned}$$

Combining the limits of II_{a1} , II_{a2} , and II_{a3} , we have

$$II_a \rightarrow_p \frac{1}{90} E(c_i^2 \theta_i^2). \quad (3)$$

Next, for III_a , write $X_{iT} = \frac{1}{T^2\sigma_i^2} \sum_{t=1}^T \bar{Y}_{it,T}(0) (\bar{Y}_{it,T} - \bar{Y}_{it,T}(0))$. Also define

$$X_{1iT} = \frac{1}{T\sigma_i^2} \sum_{t=1}^T \begin{bmatrix} \left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T} \right) u_{iq} \right) \\ - \left(\frac{t}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T} \right) u_{iq} \right) \\ - \left(\frac{t}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T} \right) u_{iq} \right) \\ + \left(\frac{t}{T} \right)^2 \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T} \right) u_{iq} \right) \end{bmatrix},$$

and

$$X_{2iT} = \frac{1}{T\sigma_i^2} \sum_{t=1}^T \begin{bmatrix} \left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-2} \left(\frac{t-q}{T} \right) \left(\frac{t-q-1}{T} \right) u_{iq} \right) \\ - \left(\frac{t}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-2} \left(\frac{T-q}{T} \right) \left(\frac{T-q-1}{T} \right) u_{iq} \right) \\ - \left(\frac{t}{T} \right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-2} \left(\frac{t-q}{T} \right) \left(\frac{t-q-1}{T} \right) u_{iq} \right) \\ + \left(\frac{t}{T} \right)^2 \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-2} \left(\frac{T-q}{T} \right) \left(\frac{T-q-1}{T} \right) u_{iq} \right) \end{bmatrix}.$$

Then, by (2), we have

$$III_a \sim -\frac{2}{n^{3/4}} \sum_{i=1}^n c_i^2 \theta_i X_{1iT} + \frac{1}{n} \sum_{i=1}^n c_i^2 \theta_i^2 X_{2iT} = -2III_{a1} + III_{a2}, \text{ say.}$$

A direct calculation shows that

$$\begin{aligned}
& EIII_{a1} \\
&= \frac{E(c_i^2 \theta_i)}{n^{3/4}} \sum_{i=1}^n EX_{1iT} \\
&= E(c_i^2 \theta_i) n^{1/4} \frac{1}{T} \sum_{t=1}^T \left[\begin{aligned} & \frac{1}{T} \sum_{p=0}^{t-1} \frac{t-p}{T} - \left(\frac{t}{T}\right) \frac{1}{T} \sum_{p=0}^{t-1} \left(\frac{T-p}{T}\right) \\ & - \left(\frac{t}{T}\right) \frac{1}{T} \sum_{p=0}^{t-1} \left(\frac{t-p}{T}\right) + \left(\frac{t}{T}\right)^2 \frac{1}{T} \sum_{p=0}^{T-1} \left(\frac{T-p}{T}\right) \end{aligned} \right] \\
&= E(c_i^2 \theta_i) n^{1/4} \int_0^1 \left(\int_0^r (r-s) ds - r \int_0^r (1-s) ds - r \int_0^r (r-s) ds + r^2 \int_0^1 (1-s) ds \right) dr \\
&\quad + O\left(\frac{n^{1/4}}{T}\right) \\
&= o(1),
\end{aligned}$$

since $\int_0^1 \left(\int_0^r (r-s) ds - r \int_0^r (1-s) ds - r \int_0^r (r-s) ds + r^2 \int_0^1 (1-s) ds \right) dr = 0$ and $\frac{n}{T} \rightarrow 0$. Also,

$$\begin{aligned}
& E(c_i^4 \theta_i^2 X_{1iT}^2) \\
&\leq \frac{2E(c_i^4 \theta_i^2)}{T\sigma_i^2} \sum_{t=1}^T \left\{ \begin{aligned} & E \left[\left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T}\right) u_{iq} \right) \right]^2 \\ & + E \left[\left(\frac{t}{T}\right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^t u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T}\right) u_{iq} \right) \right]^2 \\ & + E \left[\left(\frac{t}{T}\right) \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{t-1} \left(\frac{t-q}{T}\right) u_{iq} \right) \right]^2 \\ & + E \left[\left(\frac{t}{T}\right)^2 \left(\frac{1}{\sqrt{T}} \sum_{p=0}^T u_{ip} \right) \left(\frac{1}{\sqrt{T}} \sum_{q=0}^{T-1} \left(\frac{T-q}{T}\right) u_{iq} \right) \right]^2 \end{aligned} \right\} \\
&= M \text{ for some finite constant } M.
\end{aligned}$$

Therefore,

$$\begin{aligned}
E(III_{a1}^2) &= Var(III_{a1}) + (E(III_{a1}))^2 \\
&\leq \frac{1}{n\sqrt{n}} \sum_{i=1}^n E(c_i^2 \theta_i) E(X_{i1T}^2) + (EI_3)^2 \\
&= O\left(\frac{1}{n^{1/2}}\right) + O\left(\frac{n^{1/2}}{T^2}\right) = o(1),
\end{aligned}$$

which yields

$$III_{a1} = o_p(1).$$

Next, by the WLLN, we have

$$\begin{aligned}
III_{a2} &\rightarrow_p E(c_i^2 \theta_i^2) \left[\begin{aligned} & \int_0^1 \int_0^r (r-s)^2 ds dr - \int_0^1 r \int_0^r (1-s)^2 ds dr \\ & - \int_0^1 r \int_0^r (r-s)^2 ds dr + \int_0^1 r^2 dr \left(\int_0^1 (1-s)^2 ds \right) \end{aligned} \right] \\
&= -\frac{1}{45} E(c_i^2 \theta_i^2).
\end{aligned}$$

Combining the limits of III_{a1} and III_{a2} , we have

$$III_a \rightarrow_p -\frac{1}{45} E(c_i^2 \theta_i^2). \quad (4)$$

Using (1), (3), and (4), we deduce the required result for Part (a). ■

Part (b): Write

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it} y_{is} h_T(t, s) - \omega_{2T} \right] \\ &= I_b + II_b, \text{ say.} \end{aligned}$$

Rewriting term I_b as

$$I_b = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left\{ \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}(0)^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it}(0) y_{is}(0) h_T(t, s) - \omega_{2T} \right\},$$

and noticing that under the assumptions in the lemma,

$$E \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}(0)^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it}(0) y_{is}(0) h_T(t, s) \right) = \omega_{2T},$$

$$\frac{1}{n} \sum_{i=1}^n \text{Var} \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}(0)^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it}(0) y_{is}(0) h_T(t, s) - \omega_{2T} \right] \rightarrow \frac{11}{6300},$$

and

$$\sup_{i, T} E \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it}(0)^2 - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T y_{it}(0) y_{is}(0) h_T(t, s) - \omega_{2T} \right]^4 < M,$$

we apply the double-indexed CLT in Phillips and Moon (1999a) and deduce that

$$I_b \Rightarrow N \left(0, \frac{11}{6300} \right). \quad (5)$$

For II_b , we further decompose the term II_b into the components

$$II_b = II_{b1} + 2II_{b2},$$

where

$$II_{b1} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T (y_{it} - y_{it}(0))^2 - \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T (y_{it} - y_{it}(0)) (y_{is} - y_{is}(0)) h_T(t, s)$$

and

$$II_{b2} = \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T (y_{it} - y_{it}(0)) y_{it}(0) - \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T (y_{it} - y_{it}(0)) y_{is}(0) h_T(t, s).$$

For II_{b1} , by (2) and by the WLLN, we have

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T (y_{it} - y_{it}(0))^2 \\ &= \frac{1}{n} \sum_{i=1}^n \theta_i^2 \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left(\sum_{p=0}^{t-1} \left(\frac{t-p}{T} \right) u_{ip} \right)^2 \right] + O_p \left(\frac{1}{n^{1/4}} \right) \\ &\rightarrow_p E(\theta_i^2) \int_0^1 \int_0^r (r-s)^2 ds dr = \frac{1}{12} E(\theta_i^2), \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T (y_{it} - y_{it}(0)) (y_{is} - y_{is}(0)) h_T(t, s) \\ &= \frac{1}{n} \sum_{i=1}^n \theta_i^2 \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{s=1}^T \sum_{p=0}^{t-1} \sum_{q=0}^{s-1} \left(\frac{t-p}{T} \right) \left(\frac{s-q}{T} \right) h_T(t, s) u_{ip} u_{iq} \\ & \quad + O_p \left(\frac{1}{n^{1/4}} \right) \\ &\rightarrow_p E(\theta_i^2) \int_0^1 \int_0^1 \int_0^{r \wedge s} (r-p)(s-p) h(r, s) dp ds dr = \frac{17}{210} E(\theta_i^2). \end{aligned}$$

Therefore,

$$II_{b1} \rightarrow_p \frac{1}{420} E(\theta_i^2). \quad (6)$$

Next, in view of (2) with $\kappa = \frac{1}{4}$, we have

$$II_{b2} = -\frac{2}{n^{3/4}} \sum_{i=1}^n \theta_i X_{1iT} + \frac{1}{n} \sum_{i=1}^n \theta_i^2 X_{2iT} + o_p(1),$$

where

$$X_{1iT} = \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \sum_{s=0}^{t-1} \sum_{q=0}^t \left(\frac{t-s}{T} \right) u_{is} u_{iq} - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{p=1}^T \sum_{s=0}^{t-1} \sum_{q=0}^p \left(\frac{t-s}{T} \right) h_T(t, p) u_{is} u_{iq},$$

and

$$\begin{aligned} X_{2iT} &= \frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \sum_{s=0}^{t-1} \sum_{q=0}^t \left(\frac{t-s}{T} \right) \left(\frac{t-s-1}{T} \right) u_{is} u_{iq} \\ & \quad - \frac{1}{T^3 \sigma_i^2} \sum_{t=1}^T \sum_{p=1}^T \sum_{s=0}^{t-1} \sum_{q=0}^p \left(\frac{t-s}{T} \right) \left(\frac{t-s-1}{T} \right) h_T(t, p) u_{is} u_{iq}. \end{aligned}$$

A direct calculation shows that

$$EX_{1iT} = \left[\frac{1}{T^2} \sum_{t=1}^T \sum_{s=1}^{t-1} \left(\frac{t-s}{T} \right) - \frac{1}{T^3} \sum_{t=1}^T \sum_{p=1}^T \sum_{s=0}^{t \wedge p - 1} \left(\frac{t-s}{T} \right) h_T(t, p) \right] = O \left(\frac{1}{T} \right),$$

because

$$EX_{1iT} - \int_0^1 \int_0^r (r-s) ds dr + \sigma^2 \int_0^1 \int_0^1 \int_0^{r \wedge p} (r-s) h(r,p) ds dp dr = O\left(\frac{1}{T}\right),$$

and

$$\int_0^1 \int_0^r (r-s) ds dr - \int_0^1 \int_0^1 \int_0^{r \wedge p} (r-s) h(r,p) ds dp dr = 0.$$

Also,

$$\begin{aligned} & \sup_i EX_{1iT}^2 \\ & \leq \sup_i \frac{2}{T^4 \sigma_i^4} \sum_{t=1}^T \sum_{x=1}^T \sum_{s=0}^{t-1} \sum_{y=0}^{x-1} \sum_{q=0}^t \sum_{z=0}^x \left(\frac{t-s}{T}\right) \left(\frac{x-y}{T}\right) E[u_{is} u_{iq} u_{iy} u_{iz}] \\ & \quad + \sup_i \frac{2}{T^6 \sigma_i^4} \sum_{t=1}^T \sum_{p=1}^T \sum_{x=1}^T \sum_{y=1}^T \sum_{s=0}^{t-1} \sum_{q=0}^{p-1} \sum_{z=0}^x \sum_{w=0}^y \left(\frac{t-s}{T}\right) \left(\frac{x-z}{T}\right) h_T(t,p) h_T(x,y) E[u_{is} u_{iq} u_{iz} u_{iw}] \\ & = O(1). \end{aligned}$$

Therefore

$$\begin{aligned} -\frac{1}{n^{3/4}} \sum_{i=1}^n \theta_i X_{1iT} &= -\frac{1}{n^{3/4}} \sum_{i=1}^n \theta_i (X_{1iT} - EX_{1iT}) + \frac{1}{n^{3/4}} \sum_{i=1}^n \theta_i (EX_{1iT}) \\ &= O_p\left(\frac{1}{n^{1/4}}\right) + O\left(\frac{n^{1/4}}{T}\right) = o_p(1). \end{aligned}$$

Next, by the WLLN, we have

$$\begin{aligned} \frac{1}{n\sigma^2} \sum_{i=1}^n \theta_i^2 X_{2iT} &\rightarrow_p E(\theta_i^2) \left[\int_0^1 \int_0^r (r-s)^2 ds dr - \int_0^1 \int_0^1 \int_0^{r \wedge p} (r-s)^2 h(r,p) ds dp dr \right] \\ &= -E(\theta_i^2) \frac{1}{210}. \end{aligned}$$

Therefore, we have

$$II_{b2} \rightarrow_p -E(\theta_i^2) \frac{1}{210}. \quad (7)$$

Combining the limits of the terms I_b , II_{b1} , and II_{b2} in (5), (6), and (7), respectively, we have the desired result for Part (b). ■

Lemma 7 *Let M be a finite constant. Under Assumptions 1 and 2, the following hold.*

$$(a) \sup_i E \left[\left(\frac{1}{T} \sum_{t=1}^T u_{it} y_{it-1} \right)^2 \right] < M.$$

$$(b) \sup_i E \left[\left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right)^2 \right] < M.$$

$$(c) \sup_i E [y_{i0}^2] < M.$$

Proof. The lemma follows by direct calculation and we omit the proof. ■

Lemma 8 *Suppose that Assumptions 1–3 and 4 hold. Then, the following hold.*

$$(a) \sum_{i=1}^n (\hat{\sigma}_{1,iT}^2 - \sigma_i^2)^2 = o_p(1).$$

$$(b) \sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \sigma_i^2| = o_p(1).$$

(c) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \hat{\sigma}_{1,iT}^2 \geq M$.*

Proof.

Part (a): The required result follows by Lemma 5(a) if $\sum_{i=1}^n (\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 = o_p(1)$. Under Assumption 4, we have

$$\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2 = \frac{y_{i0}^2}{T} + \frac{\theta_i^2}{nT} \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right) - 2 \frac{\theta_i}{n^{1/2}T} \left(\frac{1}{T} \sum_{t=1}^T u_{it} y_{it-1} \right).$$

Then, by Lemma 7 and since the support of θ_i is uniformly bounded, we have for some constant M ,

$$\begin{aligned} & \sum_{i=1}^n E (\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 \\ & \leq M \left[\frac{1}{T^2} \sum_{i=1}^n E (y_{i0}^4) + \frac{1}{n^2 T^2} \sum_{i=1}^n E \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right)^2 + \frac{1}{n T^2} \sum_{i=1}^n \left(\frac{1}{T} \sum_{t=1}^T u_{it} y_{it-1} \right)^2 \right] \\ & = O\left(\frac{n}{T^2}\right) + O\left(\frac{1}{n T^2}\right) + O\left(\frac{1}{T^2}\right) = o(1), \end{aligned} \quad (8)$$

as required. ■

Parts (b) and (c): Since $\inf_{1 \leq i \leq n} \hat{\sigma}_{1,iT}^2 \leq \inf_i \sigma_i^2 - \sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \sigma_i^2|$, under Assumption 1, the required results follow if we show that

$$\sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \sigma_i^2| = o_p(1),$$

which follows by Lemma 5(b) if we show that

$$\sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| = o_p(1).$$

This follows since for any $\varepsilon > 0$,

$$P \left\{ \sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| > \varepsilon \right\} \leq \sum_{i=1}^n P \{ |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| > \varepsilon \} \leq \frac{1}{\varepsilon^2} \sum_{i=1}^n E \left[(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 \right] \rightarrow 0$$

by (8) ■

Lemma 9 *Let M be a finite constant. Under Assumptions 1 and 2, the following hold.*

- (a) $\sup_i E \left[\left(\frac{y_{iT} - y_{i0}}{\sqrt{T}} \right)^2 \right] < M.$
- (b) $\sup_i E \left[\left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} \right)^2 \right] < M.$

Proof. The lemma follows by direct calculation, and its proof is omitted. ■

Lemma 10 *Suppose that Assumptions 1 – 3, and 4 hold. Then, the following hold.*

- (a) $\sup_{1 \leq i \leq n} (\hat{\sigma}_{2,iT}^2 - \sigma_i^2) = o_p(1).$
- (b) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \hat{\sigma}_{2,iT}^2 \geq M.$*

Proof.

Part (a): The required result follows by Lemmas 5(b) and 8(b) and the triangle inequality, if we show that $\sup_{1 \leq i \leq n} |\hat{\sigma}_{2,iT}^2 - \hat{\sigma}_{1,iT}^2| = o_p(1)$ because $\hat{\sigma}_{1,iT}^2 = \frac{1}{T} \sum_{t=1}^T (\Delta y_{it})^2$. By definition, $\sup_{1 \leq i \leq n} |\hat{\sigma}_{2,iT}^2 - \hat{\sigma}_{1,iT}^2| = \sup_{1 \leq i \leq n} \frac{y_{i0}^2}{T}$. Then, by the Markov's inequality, for any $\varepsilon > 0$,

$$P \left\{ \sup_{1 \leq i \leq n} \frac{y_{i0}^2}{T} > \varepsilon \right\} \leq \sum_{i=1}^n P \left\{ \frac{y_{i0}^2}{T} > \varepsilon \right\} \leq \frac{1}{\varepsilon T} \sum_{i=1}^n E y_{i0}^2 = O \left(\frac{n}{T} \right) = o(1). \quad \blacksquare$$

Part (b): Since $\inf_{1 \leq i \leq n} \hat{\sigma}_{2,iT}^2 \leq \inf_i \sigma_i^2 - \sup_{1 \leq i \leq n} |\hat{\sigma}_{2,iT}^2 - \sigma_i^2|$ under Assumption 1, the required result follows by Part (a). ■

Lemma 11 Under Assumptions 1–3, 6, and 7,

$$\begin{aligned}
& V_{fe2,nT}(\mathbb{C}) \\
= & \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[\frac{2}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \left(\frac{y_{i0}}{\sqrt{T}} \right)^2 + \sigma_i^2 \right] \\
& + \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \left[\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - 2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) \right. \\
& \quad \left. + \frac{1}{3} \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \sigma_i^2 \omega_{p2T} \right] \\
& + \frac{1}{n} \sum_{i=1}^n \frac{c_i^4}{\sigma_i^2} \left[- \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right)^2 + \frac{2}{3} \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) \right. \\
& \quad \left. - \frac{1}{9} \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \sigma_i^2 \omega_{p4T} \right] \\
& + \frac{1}{n^{1/4} T} \sum_{i=1}^n \frac{\mathcal{S}_{1iT}}{\sigma_i^2} + \frac{1}{n^{1/2} T^{1/2}} \sum_{i=1}^n \frac{\mathcal{S}_{2iT}}{\sigma_i^2} + \frac{1}{n^{5/4}} \sum_{i=1}^n \frac{\mathcal{S}_{3iT}}{\sigma_i^2},
\end{aligned}$$

with $\frac{1}{n} \sum_{i=1}^n E[\mathcal{S}_{kiT}^2] = O(1)$, for $k = 1, 2, 3$ when $(n, T \rightarrow \infty)$ with $\frac{n}{T} \rightarrow 0$.

Proof: By definition,

$$\begin{aligned}
V_{fe2,nT}(\mathbb{C}) &= V_{fe21,nT}(\mathbb{C}) + V_{fe22,nT}(\mathbb{C}) \\
&+ \left(\frac{1}{n^{1/4}} \sum_{i=1}^n c_i \right) + \left(\frac{1}{n^{1/2}} \sum_{i=1}^n c_i^2 \right) \left(-\frac{1}{T} \sum_{t=2}^T \frac{t-1}{T} + 2 \frac{1}{T} \sum_{t=2}^T \left(\frac{t-1}{T} \right)^2 - \frac{1}{3} \right) \\
&+ \left(\frac{1}{n} \sum_{i=1}^n c_i^4 \right) \left(\frac{1}{T^2} \sum_{t=2}^T \sum_{s=2}^T \frac{t-1}{T} \frac{s-1}{T} \min \left(\frac{t-1}{T}, \frac{s-1}{T} \right) - \frac{2}{3} \left(\frac{1}{T} \sum_{t=2}^T \left(\frac{t-1}{T} \right)^2 \right) + \frac{1}{9} \right),
\end{aligned}$$

where

$$\begin{aligned}
V_{fe21,nT}(\mathbb{C}) &= \sum_{i=1}^n \frac{1}{\sigma_i^2} [(\Delta_{c_i} \underline{Y}_i)' (\Delta_{c_i} \underline{Y}_i) - (\Delta \underline{Y}_i)' (\Delta \underline{Y}_i)] \\
&= \frac{2}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(\frac{1}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} \right) + \frac{1}{n^{1/2} T^2} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \sum_{t=1}^T y_{it-1}^2 \\
V_{fe22,nT}(\mathbb{C}) &= \sum_{i=1}^n \frac{1}{\sigma_i^2} \left[\begin{array}{c} (\Delta \underline{Y}_i)' \Delta G (\Delta G' \Delta G)^{-1} \Delta G' (\Delta \underline{Y}_i) \\ - (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} G (\Delta_{c_i} G' \Delta_{c_i} G)^{-1} \Delta_{c_i} G' (\Delta_{c_i} \underline{Y}_i) \end{array} \right].
\end{aligned}$$

Let $D = \text{diag}(\sqrt{T}, 1)$ and $\tilde{G} = GD$. Then,

$$\begin{aligned}
& V_{fe22,nT}(\mathbb{C}) \\
&= \sum_{i=1}^n \frac{1}{\sigma_i^2} \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right) \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \\
&\quad - \sum_{i=1}^n \frac{1}{\sigma_i^2} \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right) \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \\
&= \sum_{i=1}^n \frac{1}{\sigma_i^2} \text{tr} \left[\begin{array}{c} \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} \\ \times \left\{ \left(\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta \underline{Y}_i)' \Delta \tilde{G} \right)' - \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right\} \end{array} \right] \\
&\quad + \sum_{i=1}^n \frac{1}{\sigma_i^2} \text{tr} \left[\left\{ \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} - \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \right\} \left(\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) \right) \left(\frac{1}{\sqrt{T}} (\Delta_{c_i} \underline{Y}_i)' \Delta_{c_i} \tilde{G} \right)' \right] \\
&= V_{fe221,nT}(\mathbb{C}) + V_{fe222,nT}(\mathbb{C}), \text{ say.}
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\
\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} &= \begin{pmatrix} 1 + \frac{c_i^2}{n^{1/2}} \frac{1}{T} & \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^T \frac{t}{T} \right) \right) \\ \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} + \frac{c_i^2}{n^{1/2}} \left(\frac{1}{T} \sum_{t=1}^T \frac{t}{T} \right) \right) & \frac{1}{T} \sum_{t=1}^T \left(1 + \frac{c_i}{n^{1/4}} \frac{t}{T} \right)^2 \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta \tilde{G}' (\Delta \underline{Y}_i) &= \begin{pmatrix} y_{i0} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) \end{pmatrix}, \\
\frac{1}{\sqrt{T}} \Delta_{c_i} \tilde{G}' (\Delta_{c_i} \underline{Y}_i) &= \begin{pmatrix} y_{i0} + \frac{c_i}{n^{1/4}} \frac{1}{T} (y_{iT} - y_{i1}) + \frac{c_i^2}{n^{1/2}} \frac{1}{T^2} \sum_{t=1}^T y_{it-1} \\ \frac{1}{\sqrt{T}} (y_{iT} - y_{i0}) + \frac{c_i}{n^{1/4}} \frac{1}{\sqrt{T}} y_{iT} + \frac{c_i^2}{n^{1/2}} \frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \end{pmatrix}.
\end{aligned}$$

Computation of $V_{fe221,nT}(\mathbb{C})$: A direct calculation shows that

$$\begin{aligned}
& V_{fe221,nT}(\mathbb{C}) \\
&= \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(2 \left(\frac{y_{i0}}{\sqrt{T}} \right)^2 - 2 \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 \right) \\
&\quad + \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \left(-2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) - \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \frac{1}{T} \mathcal{R}_{2iT} \right) \\
&\quad + \frac{1}{n^{3/4}} \sum_{i=1}^n \frac{c_i^3}{\sigma_i^2} \left(-2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) + \frac{1}{T} \mathcal{R}_{3iT} \right) \\
&\quad + \frac{1}{n} \sum_{i=1}^n \frac{c_i^4}{\sigma_i^2} \left(- \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right)^2 + \frac{1}{T} \mathcal{R}_{4iT} \right),
\end{aligned}$$

where $\frac{1}{n} \sum_{i=1}^n E \mathcal{R}_{kiT}^2 = O(1)$ for $k = 2, 3, 4$.

Computation of $V_{fe222,nT}(\mathbb{C})$:

Direct calculation gives

$$\begin{aligned} & \left(\frac{1}{T} \Delta \tilde{G}' \Delta \tilde{G} \right)^{-1} - \left(\frac{1}{T} \Delta_{c_i} \tilde{G}' \Delta_{c_i} \tilde{G} \right)^{-1} \\ = & \begin{pmatrix} 0 & \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} - \frac{1}{2} \frac{c_i^2}{n^{1/2}} + \frac{1}{6} \frac{c_i^3}{n^{3/4}} \right) \\ \frac{1}{\sqrt{T}} \left(\frac{c_i}{n^{1/4}} - \frac{1}{2} \frac{c_i^2}{n^{1/2}} + \frac{1}{6} \frac{c_i^3}{n^{3/4}} \right) & \frac{c_i}{n^{1/4}} \left(2 \left(\frac{1}{T} \sum_{t=1}^T \frac{t}{T} \right) \right) - \frac{2}{3} \frac{c_i^2}{n^{1/2}} + \frac{1}{3} \frac{c_i^3}{n^{3/4}} - \frac{1}{9} \frac{c_i^4}{n} \end{pmatrix} \\ & + \xi_{1,i,nT} + \xi_{2,i,nT}, \end{aligned}$$

where $\xi_{1,i,nT} = O\left(\frac{1}{n^{1/2}T}\right)$ and $\xi_{2,i,nT} = O\left(\frac{1}{n^{5/4}}\right)$ uniformly across i because the support of c_i' s is bounded. Then,

$$\begin{aligned} & V_{fe222,nT}(\mathbb{C}) \\ = & \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(\left(\frac{y_{iT}}{\sqrt{T}} \right)^2 - \left(\frac{y_{i0}}{\sqrt{T}} \right)^2 \right) + \frac{1}{n^{1/4}T} \sum_{i=1}^n \frac{\mathcal{R}_{5iT}}{\sigma_i^2} \\ & + \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \frac{4}{3} \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \frac{1}{n^{1/2}T^{1/2}} \sum_{i=1}^n \frac{\mathcal{R}_{6iT}}{\sigma_i^2} \\ & + \frac{1}{n^{3/4}} \sum_{i=1}^n \frac{c_i^3}{\sigma_i^2} 2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) + \frac{1}{n^{3/4}T^{1/2}} \sum_{i=1}^n \frac{\mathcal{R}_{7iT}}{\sigma_i^2} \\ & + \frac{1}{n} \sum_{i=1}^n \frac{c_i^4}{\sigma_i^2} \left(\frac{2}{3} \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) - \frac{1}{9} \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 \right) + \frac{1}{nT^{1/2}} \sum_{i=1}^n \frac{\mathcal{R}_{8iT}}{\sigma_i^2} \\ & + \frac{1}{n^{5/4}} \sum_{i=1}^n \frac{\mathcal{R}_{9iT}}{\sigma_i^2}, \end{aligned}$$

where $\frac{1}{n} \sum_{i=1}^n E \mathcal{R}_{kiT}^2 = O(1)$ for $k = 5, \dots, 9$. Putting the terms in $V_{fe21,nT}(\mathbb{C})$, $V_{fe221,nT}(\mathbb{C})$, and $V_{fe222,nT}(\mathbb{C})$ together with the restriction that $n, T \rightarrow \infty$ with $\frac{n}{T} \rightarrow 0$, we have the required result. ■

Lemma 12 *Under Assumptions 1 – 3, 6, and 7, the following hold:*

- (a) $\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[\frac{2}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \left(\frac{y_{i0}}{\sqrt{T}} \right)^2 + \sigma_i^2 \right] = o_p(1)$;
- (b) $\frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \left[\begin{aligned} & \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - \sigma_i^2 \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) + \frac{1}{3} \left\{ \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 - \sigma_i^2 \right\} \\ & - \left\{ 2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) - \sigma_i^2 \frac{2}{T} \sum_{t=1}^T \left(\frac{t}{T} \right) \left(\frac{t-1}{T} \right) \right\} \end{aligned} \right] \Rightarrow$
 $N\left(-\frac{1}{90} E(c_i^2 \theta_i^2), \frac{1}{45} E(c_i^4)\right)$;
- (c) $\frac{1}{n} \sum_{i=1}^n \frac{c_i^4}{\sigma_i^2} \left[\begin{aligned} & - \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right)^2 + \frac{2}{3} \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t}{T} y_{it-1} \right) \\ & - \frac{1}{9} \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 + \sigma_i^2 \omega_{p4T} \end{aligned} \right] =$
 $o_p(1)$.

Proof:

Part (a): First, notice from

$$y_{it}^2 - y_{it-1}^2 = (\rho_i^2 - 1) y_{it-1}^2 + 2\rho_i y_{it-1} u_{it} + u_{it}^2 \text{ for } t \geq 1,$$

that

$$\left(\frac{y_{iT}}{\sqrt{T}}\right)^2 - \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 = (\rho_i^2 - 1) \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + 2\rho_i \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} + \frac{1}{T} \sum_{t=1}^T u_{it}^2.$$

Since $\Delta y_{it} = (\rho_i - 1) y_{it-1} + u_{it}$, we have

$$2\frac{1}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} = 2(\rho_i - 1) \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it}.$$

Then,

$$\begin{aligned} & \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[2\frac{1}{T} \sum_{t=1}^T \Delta y_{it} y_{it-1} - \left(\frac{y_{iT}}{\sqrt{T}}\right)^2 + \left(\frac{y_{i0}}{\sqrt{T}}\right)^2 + \sigma_i^2 \right] \\ &= \frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left[-(\rho_i - 1)^2 \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 + 2(1 - \rho_i) \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} - \left(\frac{1}{T} \sum_{t=1}^T u_{it}^2 - \sigma_i^2\right) \right]. \end{aligned}$$

Under the assumptions of the lemma,

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} (\rho_i - 1)^2 \frac{1}{T} \sum_{t=1}^T y_{it-1}^2 = \frac{n^{1/4}}{T} \left(\frac{1}{n} \sum_{i=1}^n c_i \theta_i^2 \left(\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T y_{it-1}^2 \right) \right) = O_p \left(\frac{n^{1/4}}{T} \right),$$

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} (1 - \rho_i) \frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} = \frac{1}{T} \frac{2}{n^{1/2}} \sum_{i=1}^n c_i \left(\frac{1}{T \sigma_i^2} \sum_{t=1}^T y_{it-1} u_{it} \right) = O_p \left(\frac{1}{T} \right),$$

and

$$\frac{1}{n^{1/4}} \sum_{i=1}^n \frac{c_i}{\sigma_i^2} \left(\frac{1}{T} \sum_{t=1}^T u_{it}^2 - \sigma_i^2 \right) = O_p \left(\frac{n^{1/4}}{T^{1/2}} \right),$$

leading to the required result for Part (a). ■

Part (b): Under the assumptions of the lemma, it is possible to show that

$$\begin{aligned} & \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{c_i^2}{\sigma_i^2} \left[\left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 - \sigma_i^2 \frac{1}{T} \sum_{t=1}^T \frac{t-1}{T} \right) + \frac{1}{3} \left\{ \left(\frac{y_{iT}}{\sqrt{T}} \right)^2 - \sigma_i^2 \right\} \right. \\ & \quad \left. - \left\{ 2 \left(\frac{y_{iT}}{\sqrt{T}} \right) \left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} y_{it-1} \right) - \sigma_i^2 \frac{2}{T} \sum_{t=1}^T \left(\frac{t-1}{T} \right)^2 \right\} \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n c_i^2 \left[\frac{1}{T^2 \sigma_i^2} \sum_{t=1}^T \left\{ (y_{it} - y_{i0}) - \frac{t}{T} (y_{iT} - y_{i0}) \right\}^2 - \omega_{1T} \right] + o_p(1) \\ &\Rightarrow N \left(-\frac{1}{90} E(c_i^2 \theta_i^2), \frac{1}{45} E(c_i^4) \right), \end{aligned}$$

where the last limit holds by Lemma 6(a). ■

Part (c): Under the assumptions of the lemma, Part (c) follows by the WLLN. ■

Lemma 13 *Let M be a finite constant. Under Assumptions 1 – 3, 6, and 7, the following hold.*

- (a) $\sup_i E [y_{i0}^4] < M.$
- (b) $\sup_i E \left[\left(\frac{y_{iT}}{\sqrt{T}} \right)^4 \right] < M.$
- (c) $\sup_i E \left[\left(\frac{1}{T} \sum_{t=1}^T y_{it-1} u_{it} \right)^2 \right] < M.$
- (d) $\sup_i E \left[\left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right)^2 \right] < M.$
- (e) $\sup_i E \left[\left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T y_{it-1} \right)^4 \right] < M.$
- (f) $\sup_i E \left[\left(\frac{1}{T\sqrt{T}} \sum_{t=1}^T \frac{t-1}{T} y_{it-1} \right)^4 \right] < M.$

Proof. The lemma follows by direct calculations and we omit the proof. ■

Lemma 14 *Suppose that Assumptions 1 – 3, and 6 hold. Then, the following hold.*

- (a) $\sum_{i=1}^n (\hat{\sigma}_{1,iT}^2 - \sigma_i^2)^2 = o_p(1).$
- (b) $\sup_{1 \leq i \leq n} (\hat{\sigma}_{1,iT}^2 - \sigma_i^2) = o_p(1).$
- (c) $\sum_{i=1}^n (\hat{\sigma}_{3,iT}^2 - \sigma_i^2)^2 = o_p(1).$
- (d) $\sup_{1 \leq i \leq n} (\hat{\sigma}_{3,iT}^2 - \sigma_i^2) = o_p(1).$
- (e) *With probability approaching one, there exists a constant $M > 0$ such that $\inf_i \hat{\sigma}_{3,iT}^2 \geq M.$*

Proof.

Part (a): The required result follows by Lemma 5(a) if $\sum_{i=1}^n (\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 = o_p(1).$ Under Assumption 6, we have

$$\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2 = \frac{y_{i0}^2}{T} + \frac{\theta_i^2}{n^{1/2}T} \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right) - 2 \frac{\theta_i}{n^{1/4}T} \left(\frac{1}{T} \sum_{t=1}^T u_{it} y_{it-1} \right).$$

Then, by Lemma 13 and since the support of θ_i is uniformly bounded, we have for some constant M ,

$$\begin{aligned}
& \sum_{i=1}^n E (\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 \\
& \leq M \left[\frac{1}{T^2} \sum_{i=1}^n E (y_{i0}^4) + \frac{1}{nT^2} \sum_{i=1}^n E \left(\frac{1}{T^2} \sum_{t=1}^T y_{it-1}^2 \right)^2 + \frac{1}{n^{1/2}T^2} \sum_{i=1}^n \left(\frac{1}{T} \sum_{t=1}^T u_{it} y_{it-1} \right)^2 \right] \\
& = O\left(\frac{n}{T^2}\right) + O\left(\frac{1}{T^2}\right) + O\left(\frac{n^{1/2}}{T^2}\right) = o(1), \tag{9}
\end{aligned}$$

as required. ■

Part (b): By the triangle inequality and Lemma 5(b), the required result follows if we show

$$\sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| = o_p(1).$$

Since for any $\varepsilon > 0$,

$$P \left\{ \sup_{1 \leq i \leq n} |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| > \varepsilon \right\} \leq \sum_{i=1}^n P \{ |\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2| > \varepsilon \} \leq \frac{1}{\varepsilon^2} \sum_{i=1}^n E \left[(\hat{\sigma}_{1,iT}^2 - \tilde{\sigma}_{iT}^2)^2 \right] \rightarrow 0$$

by (9), we have the required result. ■

Part (c): The required result follows by Part (a) and the inequality $(a+b)^2 \leq 2a^2 + 2b^2$, if we show that

$$\sum_{i=1}^n (\hat{\sigma}_{3,iT}^2 - \hat{\sigma}_{1,iT}^2)^2 = o_p(1),$$

for which it is enough by the Markov inequality to show that

$$\sum_{i=1}^n E (\hat{\sigma}_{3,iT}^2 - \hat{\sigma}_{1,iT}^2)^2 \rightarrow 0. \tag{10}$$

Notice by definition that

$$\begin{aligned}
\hat{\sigma}_{3,iT}^2 &= \hat{\sigma}_{1,iT}^2 - \frac{1}{T} (\Delta \underline{Y}_i)' \Delta G (\Delta G' \Delta G)^{-1} \Delta G' (\Delta \underline{Y}_i) \\
&= \hat{\sigma}_{1,iT}^2 - \frac{1}{T} \left(y_{i0}^2 + \left(\frac{y_{iT} - y_{i0}}{\sqrt{T}} \right)^2 \right).
\end{aligned}$$

Then the required result (10) follows by Lemma 13(a) and (b). ■

Part (d): The required result follows by Part (b) and the triangle inequality, if we show

$$\sup_{1 \leq i \leq n} |\hat{\sigma}_{3,iT}^2 - \hat{\sigma}_{1,iT}^2| = o_p(1),$$

which follows since $\sup_{1 \leq i \leq n} |\hat{\sigma}_{3,iT}^2 - \hat{\sigma}_{1,iT}^2|^2 \leq \sum_{i=1}^n (\hat{\sigma}_{3,iT}^2 - \hat{\sigma}_{1,iT}^2)^2$ and by (10). ■

Part (e): Since $\inf_{1 \leq i \leq n} \hat{\sigma}_{3,iT}^2 \leq \inf_i \sigma_i^2 - \sup_{1 \leq i \leq n} |\hat{\sigma}_{3,iT}^2 - \sigma_i^2|$ under Assumption 1, the required result follows by Part (d). ■