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On preconditioners for mortar discretization of elliptic problems

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SUMMARY

We consider elliptic problems with discontinuous coefficients defined on a union of two polygonal subdomains. The problems are discretized by the finite element method on non-matching triangulation across the interface. The discrete problems are described by the mortar technique in the space with constraints (the mortar condition) and in the space without constraints using Lagrange multipliers. To solve the discrete problems Preconditioned conjugate gradient iterations are used with Neumann–Dirichlet and Neumann–Neumann preconditioners in the first case, and dual Neumann–Dirichlet and dual Neumann–Neumann (or FETI, the finite element tearing and interconnecting) in the second case. An analysis of convergence of all four of these preconditioners is given. Numerical comparison of their performance on non-matching grids is presented. The general observation is that all preconditioners considered are very robust for the cases with the discontinuity ratio of 1000 across the interface.

KEY WORDS: domain decomposition; mortar finite element method; non-matching grids; saddle-point problem; elliptic problems with discontinuous coefficients

1. INTRODUCTION

In this paper, we discuss a second-order elliptic problem with discontinuous coefficients defined on a polygonal region Ω which is a union of two Ω_i polygons. The problem is discretized by the finite element method on non-matching triangulation across $\bar{\Gamma} = \bar{\Omega}_1 \cap \bar{\Omega}_2$. The discrete problem is described using the mortar technique, see References [1, 2].

The goal of this paper is to compare four preconditioners used for solving the discrete problem with jumps of the coefficients at the interface formulated in the space with constraints and

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without constraints using Lagrange multipliers. For the former formulation Neumann–Dirichlet (N–D) and Neumann–Neumann (N–N) preconditioners are discussed. They are known for continuous coefficients and also for many substructures, see References [3, 4], the book [5] and references therein. For the latter formulation preconditioners dual to N–N and dual to N–N (FETI) are discussed. Such algorithms designed and analysed on matching grids are described in References [6–8].

Analysis of convergence of the discussed preconditioners is given. For the dual preconditioners such analysis to our knowledge has not yet been previously established. It is proved that the rate of convergence of the discussed preconditioners is independent of the mesh triangulation and the jump of coefficients. The theory is supported by numerical experiments which confirm the theoretical results.

Several preconditioners different from the discussed in this paper have been designed and analysed in the literature for the mortar discrete problem, see References [5, 9, 11] and references therein. Most of them are for problems with continuous coefficients.

The paper is organized as follows. In Section 2, the differential and discrete problems are formulated. In Section 3, a matrix form of discrete problems is given. The preconditioners are described and analysed in Sections 4 and 5, while some aspects of their implementation are presented in Section 6. Finally, numerical results and comparisons of the considered preconditioners are given in Section 7.

2. MORTAR DISCRETE PROBLEM

We consider the following differential problem:

Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega) \tag{1}$$

where

$$a(u, v) = (\rho(x)\nabla u, \nabla v)_{L^2(\Omega)}, \quad f(v) = (f, v)_{L^2(\Omega)}$$

We assume that Ω is a polygonal region. Let Ω be a union of two disjoint polygonal subregions Ω_i, $i = 1, 2$, of a diameter one. We additionally assume that $\rho(x) \geqslant \rho_0 > 0$ is a continuous function in each Ω_i and, for simplicity of presentation, that $\rho(x) = \rho_i = \text{constant}$ on Ω_i.

In each Ω_i, a triangulation is introduced with triangular elements $e_i^{(k)}$ and a parameter $h_i = \max_k h_i^{(k)}$, where $h_i^{(k)}$ is a diameter of $e_i^{(k)}$. The resulting triangulation of Ω is non-matching across $\bar{\Gamma} = \bar{\Omega}_1 \cap \bar{\Omega}_2$. We assume that the h_i-triangulation in each Ω_i is quasi-uniform, see Reference [10].

Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions defined on the triangulation of Ω_i and vanishing on $\partial\Omega_i \cap \partial\Omega$, and let

$$X^h(\Omega) = X_1(\Omega_1) \times X_2(\Omega_2)$$

Note that $X^h \not\subset H_0^1(\Omega)$; therefore it cannot be used for discretization of (1). To discretize (1) some weak continuity on Γ for $v \in X^h$ is imposed and it is called a *mortar condition*, see Reference [1]. To describe the mortar condition we assume that $\rho_1 \leqslant \rho_2$ and select a face of Ω_2, geometrically equal to Γ, as a *mortar* (*master*) and denote it by γ, while $\delta = \Gamma$ as a face of Ω_1 as *non-mortar* (*slave*). This choice is arbitrary in the case $\rho_1 = \rho_2$, however in our case,

$\rho_1 \leqslant \rho_2$ and it is important for the analysis of convergence to choose as the mortar side the one where the coefficient is larger. In the analysis of the N–N preconditioner and the finite element tearing and interconnecting (FETI) method we need that h_γ/h_δ be uniformly bounded, where h_δ and h_γ are the steps of triangulation on δ and γ, respectively.

Let $W_1(\delta)$ and $W_2(\gamma)$ be the restrictions of $X_1(\Omega_1)$ and $X_2(\Omega_2)$ to δ and γ, respectively. Note that they are different because they are defined on different 1-D triangulations of Γ. Let $M(\delta)$ be a space of piecewise linear continuous functions defined on the triangulation of δ with constant values on elements which intersect $\partial\delta$.

We say that $u=(u_1,u_2)\in X^h(\Omega)$ satisfies the mortar condition on $\delta(\delta=\gamma=\Gamma)$ if

$$\int_\delta (u_1-u_2)\psi\,\mathrm{d}s = 0, \quad \psi\in M(\delta) \tag{2}$$

Note that (2) for a given u_2 can be written as $u_1=\pi(u_2)$ where $\pi(u_2): W_2(\gamma)\to W_1(\delta)$ is defined by

$$\int_\delta \pi(u_2)\psi\,\mathrm{d}s = \int_\delta u_2\psi\,\mathrm{d}s, \quad \psi\in M(\delta) \tag{3}$$

Here $\pi(u_2)=u_1=u_2=0$ on $\partial\delta$.

Let $V^h(\Omega)$ be a subspace of $X^h(\Omega)$ of functions which satisfy the mortar condition (2) on δ. The discrete problem for (1) in $V^h(\Omega)$ is of the form:

Find $u_h^*=(u_{1h}^*,u_{2h}^*)\in V^h(\Omega)$ such that

$$\sum_{i=1}^{2} a_i(u_{ih}^*, v_{ih}) = f(v_h), \quad v_h=(v_{1h},v_{2h})\in V^h(\Omega) \tag{4}$$

where $a_i(u_i,v_i)=\rho_i(\nabla u_i,\nabla v_i)_{L^2(\Omega_i)}$. This problem has a unique solution and its error bound is known, see Reference [1].

The discrete problem (4) can be rewritten as a saddle-point problem using Lagrange multipliers as follows, see for example References [2] or [5]:

Let for $u=(u_1,u_2)\in X^h(\Omega)$ and $\psi\in M(\delta)$

$$b(u,\psi)\equiv \int_\delta (u_1-u_2)\psi\,\mathrm{d}x$$

Find $(u_h^*,\lambda_h^*)\in X^h(\Omega)\times M(\delta)$ such that

$$\begin{aligned} a(u_h^*,v_h)+b(v_h,\lambda_h^*) &= f(v_h), \quad v_h\in X^h(\Omega)\\ b(u_h^*,\psi) &= 0, \quad \psi\in M(\delta) \end{aligned} \tag{5}$$

It is easy to see that (5) is equivalent to (4), i.e. the solution u_h^* of (5) is the solution of (4) and vice versa. Therefore problem (5) has a unique solution. An analysis of (5) can be done straightforwardly using the inf–sup condition, including the error bound, see References [2, 11].

3. MATRIX FORMS

In this section, we derive the matrix forms of the discrete problems (4) and (5).

3.1. *Matrix form of* (4)

To establish a matrix form of (4) we define basis functions of V^h as follows:

Let $\varphi_k^{(i)}$, $i=1,2$, be standard nodal basis functions of $X_i(\Omega_i)$. Let $\Omega_{ih}, \delta_h, \gamma_h$ denote sets of interior nodal points of Ω_i, δ and γ, respectively, and N is the number of nodal points of $\Omega_1 \cup \Omega_2 \cup \gamma$. Basis functions $\phi_k \in V^h$, $k=1,\ldots,N$ are of the form:

$$\phi_k = \{\varphi_k^{(1)} \text{on } \bar{\Omega}_1, 0 \text{ on } \bar{\Omega}_2\}, \quad x_k \in \Omega_{1h}$$
$$\phi_k = \{0 \text{ on } \bar{\Omega}_1, \varphi_k^{(2)} \text{ on } \bar{\Omega}_2\}, \quad x_k \in \Omega_{2h}$$
$$\phi_k = \{\pi(\varphi_k^{(2)}) \text{ on } \bar{\Omega}_1,\ \varphi_k^{(2)} \text{ on } \bar{\Omega}_2\}, \quad x_k \in \gamma_h$$

where $\pi(\varphi_k^{(2)})$, defined on δ by (3), is extended by zero at $x \in \Omega_{1h}$. Note that there are no basis functions associated with $x_k \in \delta_h$. Hence $V^h = \text{span}\{\phi_k\}$, $k=1,\ldots,N$.

Using these basis functions one can rewrite (4) as

$$A\underline{u}_h^* = \underline{f}_h \tag{6}$$

where $A = \{\sum_{i=1}^{2} a_i(\phi_k, \phi_l)\}$ for $k,l=1,\ldots,N$. The matrix A is symmetric and positive definite and its condition number is of the order of h^{-2}, where $h = \min(h_1, h_2)$.

3.2. *Matrix form of* (5)

To provide a matrix form of (5) we need a matrix formulation of the mortar condition, i.e. the matrix form of $b(\cdot,\cdot)$. Using the nodal basis functions, $\varphi_k^{(1)} \in W_1(\delta)$, $\varphi_k^{(2)} \in W_2(\gamma)$, and $\psi_l \in M(\delta)$, one can rewrite Equation (2) as

$$B_\delta \underline{u}_{1\delta} - B_\gamma \underline{u}_{2\gamma} = 0 \tag{7}$$

where $\underline{u}_{1\delta}$ and $\underline{u}_{2\gamma}$ are vectors that represent $u_{1|\delta} \in W_1(\delta)$ and $u_{2|\gamma} \in W_2(\gamma)$, respectively, and

$$B_\delta = \{(\psi_l, \varphi_k^{(1)})_{L^2(\delta)}\}, \quad l,k=1,\ldots,n_\delta$$
$$B_\gamma = \{(\psi_l, \varphi_k^{(2)})_{L^2(\gamma)}\}, \quad l=1,\ldots,n_\delta;\ k=1,\ldots,n_\gamma$$

Here $n_\delta = \dim(M(\delta)) = \dim(W_1(\delta))$, $n_\gamma = \dim(W_2(\gamma))$. Note that B_δ is a square tridiagonal matrix $n_\delta \times n_\delta$, symmetric and positive definite, and $\text{cond}(B_\delta) \sim 1$, while B_γ is a rectangular matrix $n_\delta \times n_\gamma$. Hence for $(u,\lambda) \in X^h(\Omega) \times M(\delta)$

$$b(u,\lambda) = (B_\delta \underline{u}_{1\delta}, \underline{\lambda})_{R^{n_\delta}} - (B_\gamma \underline{u}_{2\gamma}, \underline{\lambda})_{R^{n_\delta}}$$

Thus (5) can be presented in the form

$$\begin{pmatrix} \tilde{A} & B^{\mathrm{T}} \\ B & 0 \end{pmatrix} \begin{pmatrix} \underline{u}^* \\ \underline{\lambda}^* \end{pmatrix} = \begin{pmatrix} \underline{F} \\ 0 \end{pmatrix} \tag{8}$$

where $(\tilde{A}\underline{u}, v)_{R^J} = a(u,v)$, J is the number of nodal points of $\Omega_1 \cup \delta \cup \Omega_2 \cup \gamma$, $\underline{u}^* = \{\underline{u}_1^*, \underline{u}_2^*\}^{\mathrm{T}}$, $\underline{u}_1^* = \{u_1(x_k)\}^{\mathrm{T}}$ for $x_k \in (\Omega_{1h} \cup \delta_h)$ and $\underline{u}_2^* = \{u_2(x_k)\}^{\mathrm{T}}$ for $x_k \in (\Omega_{2h} \cup \gamma_h)$, and

$\tilde{A} = \text{diag}\ \{A^{(1)}, A^{(2)}\}$,

$$A^{(1)} = \{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\}, \quad x_k, x_l \in (\Omega_{1h} \cup \delta_h)$$
$$A^{(2)} = \{a_2(\varphi_k^{(2)}, \varphi_l^{(2)})\}, \quad x_k, x_l \in (\Omega_{2h} \cup \gamma_h)$$

The matrix B is of the form

$$B = ([0 \quad B_\delta] \quad [0 \quad -B_\gamma]) \tag{9}$$

System (8) can be rewritten as

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} & 0 & 0 & 0 \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} & 0 & 0 & B_\delta \\ 0 & 0 & A_{II}^{(2)} & A_{I\gamma}^{(2)} & 0 \\ 0 & 0 & A_{\gamma I}^{(2)} & A_{\gamma\gamma}^{(2)} & -B_\gamma^{\mathrm{T}} \\ 0 & B_\delta & 0 & -B_\gamma & 0 \end{pmatrix} \begin{pmatrix} u_I^{(1)} \\ u_\delta^{(1)} \\ u_I^{(2)} \\ u_\gamma^{(2)} \\ \lambda_\delta \end{pmatrix} = \begin{pmatrix} F_I^{(1)} \\ F_\delta^{(1)} \\ F_I^{(2)} \\ F_\gamma^{(2)} \\ 0 \end{pmatrix} \tag{10}$$

Here $\underline{u}_1^* = \{u_I^{(1)}, u_\delta^{(1)}\}^{\mathrm{T}}$ and $\underline{u}_2^* = \{u_I^{(2)}, u_\gamma^{(2)}\}^{\mathrm{T}}$ correspond to the nodal values at the interior nodal points of Ω_i, δ and γ, respectively, and $\lambda_\delta \equiv \underline{\lambda}^*$;

$$\begin{aligned} A_{II}^{(1)} &= \{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\} \quad x_k, x_l \in \Omega_{1h} \\ A_{I\delta}^{(1)} &= \{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\} \quad x_k \in \Omega_{1h} \quad \text{and} \quad x_l \in \delta_h \\ A_{\delta\delta}^{(1)} &= \{a_1(\varphi_k^{(1)}, \varphi_l^{(1)})\} \quad x_k, x_l \in \delta_h \end{aligned}$$

$A_{II}^{(2)}$, $A_{I\gamma}^{(2)}$ and $A_{\gamma\gamma}^{(2)}$ are defined in a similar way. Note that $(A_{I\delta}^{(1)}) = (A_{\delta I}^{(1)})^{\mathrm{T}}$ and $(A_{I\gamma}^{(2)}) = (A_{\gamma I}^{(2)})^{\mathrm{T}}$. The matrix of (10) is invertible.

4. PRECONDITIONERS FOR (4)

In this section, we define and analyse preconditioners for problem (4). They will be defined for Schur complement systems with respect to unknowns corresponding to nodal points of γ.

We consider a matrix form of (4), i.e. (6),

$$Au_h^* = f_h \tag{11}$$

(below the underlining of vectors is dropped) and rewrite it as a 3 by 3 block matrix using (10).

We first eliminate the unknowns λ_δ and then $u_\delta^{(1)}$ from (10). Using rows 2 and 5 of (10) we get system (6) with the matrix A which is defined explicitly by submatrices of $A^{(1)}, A^{(2)}$ and by B_δ and B_γ, i.e. by the standard nodal basis functions.

Thus (10) can be rewritten in the form

$$Au \equiv \begin{pmatrix} A_{II}^{(1)} & 0 & K_{I\gamma} \\ 0 & A_{II}^{(2)} & A_{I\gamma}^{(2)} \\ (K_{I\gamma})^{\mathrm{T}} & A_{\gamma I}^{(2)} & K_{\gamma\gamma} \end{pmatrix} \begin{pmatrix} u_I^{(1)} \\ u_I^{(2)} \\ u_\gamma^{(2)} \end{pmatrix} = \begin{pmatrix} F_I^{(1)} \\ F_I^{(2)} \\ B_\gamma^{\mathrm{T}} B_\delta^{-1} F_\delta^{(1)} + F_\gamma^{(2)} \end{pmatrix} \tag{12}$$

where the rows correspond to $x_k \in \Omega_{1h}$, $x_k \in \Omega_{2h}$ and $x_k \in \gamma_h$, respectively, and

$$K_{I\gamma} = A_{I\delta}^{(1)} B_\delta^{-1} B_\gamma, \quad K_{\gamma\gamma} = A_{\gamma\gamma}^{(1)} + B_\gamma^{\mathrm{T}} B_\delta^{-1} A_{\delta\delta}^{(1)} B_\delta^{-1} B_\gamma \tag{13}$$

This form of A is used usually in the computations.

Next we eliminate from this system the unknowns $u_I^{(1)}$ and $u_I^{(2)}$. Using rows 1 and 2 from (12) and substituting them in the third row, we reduce (11) to

$$Su_\gamma^{(2)} = F_\gamma \tag{14}$$

where

$$S = K_{\gamma\gamma} - (K_{I\gamma})^{\mathrm{T}} (A_{II}^{(1)})^{-1} K_{I\gamma} - (A_{I\gamma}^{(2)})^{\mathrm{T}} (A_{II}^{(2)})^{-1} A_{I\gamma}^{(2)} \tag{15}$$

$$F_\gamma = B_\gamma^{\mathrm{T}} B_\delta^{-1} F_\delta^{(1)} + F_\gamma^{(2)} - (K_{I\gamma})^{\mathrm{T}} (A_{II}^{(1)})^{-1} F_I^{(1)} - (A_{I\gamma}^{(2)})^{\mathrm{T}} (A_{II}^{(2)})^{-1} F_I^{(2)} \tag{16}$$

$F_\delta^{(1)} = \{(f, \varphi_l^{(1)})\}$, $x_l \in \delta_h$ and $F_\gamma^{(2)} = \{(f, \varphi_l^{(2)})\}$, $x_l \in \gamma_h$. The Schur complement matrix S is symmetric and positive definite, n_γ by n_γ.

The matrix S can be rewritten using the standard Schur complement matrices for problems defined on $\bar{\Omega}_1$ and $\bar{\Omega}_2$. Let, see (10),

$$A^{(1)} = \begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} A_{II}^{(2)} & A_{I\gamma}^{(1)} \\ A_{\gamma I}^{(2)} & A_{\gamma\gamma}^{(2)} \end{pmatrix}$$

Their Schur complement matrices with respect to $u_\delta^{(1)}$ and $u_\gamma^{(2)}$, respectively, are of the form

$$S_1 = A_{\delta\delta}^{(1)} - A_{\delta I}^{(1)} (A_{II}^{(1)})^{-1} A_{I\delta}^{(1)}, \quad S_2 = A_{\gamma\gamma}^{(2)} - A_{\gamma I}^{(2)} (A_{II}^{(2)})^{-1} A_{I\gamma}^{(2)} \tag{17}$$

Straightforwardly one can verify that

$$S = B_\gamma^{\mathrm{T}} B_\delta^{-1} S_1 B_\delta^{-1} B_\gamma + S_2 \tag{18}$$

Additionally, we denote $\hat{S}_i = S_i$ for $\rho_i = 1$, $i = 1, 2$.

4.1. *Neumann–Dirichlet preconditioner*

The Neumann–Dirichlet (N–D) preconditioner for S is defined by S_2, see References [3, 5].

Theorem 4.1
For $v_\gamma^{(2)} \in R^{n_\gamma}$ and $\rho_1 \leqslant \rho_2$ the following holds

$$(S_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant (S v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant C(S_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{19}$$

where C is a positive constant independent of h_i and the coefficients ρ_i, $i=1,2$.

Proof
The left-hand side inequality is obvious since $S_1 = S_1^{\mathrm{T}} > 0$. The right-hand side follows from Lemma 4.1 and the assumption that $\rho_1 \leqslant \rho_2$. □

Lemma 4.1
For $v_\gamma^{(2)} \in R^{n_\gamma}$

$$(S_1 B_\delta^{-1} B_\gamma v_\gamma^{(2)}, B_\delta^{-1} B_\gamma v_\gamma^{(2)})_{R^{n_\delta}} \leqslant C(S_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{20}$$

provided that $\rho_1 \leqslant \rho_2$ where C is a positive constant independent of h_i and the coefficients ρ_i, $i=1,2$.

Proof
Note that

$$(S_1 v_\delta^{(1)}, v_\delta^{(1)})_{R^{n_\delta}} = a_1(v_1, v_1) \tag{21}$$

Here $v_1 = (v_I^{(1)}, v_\delta^{(1)})$ and $v_I^{(1)} = H_1 v_\delta^{(1)}$ is the discrete harmonic extension of $v_\delta^{(1)}$ given on δ and zero on $\partial\Omega_1 \backslash \delta$ in the sense of $(\nabla\cdot, \nabla\cdot)_{L^2(\Omega_1)}$, i.e.

$$(\nabla H_1 v_\delta^{(1)}, \nabla w_1)_{L^2(\Omega_1)} = 0, \quad w_1 \in \overset{o}{X_1}(\Omega_1) \tag{22}$$

with $H_1 v_\delta^{(1)} = v_\delta^{(1)}$ on δ and $H_1 v_\delta^{(1)} = 0$ on $\partial\Omega_1 \backslash \delta$, where $\overset{o}{X_1}(\Omega_1)$ is a space of functions of $X_1(\Omega_1)$ which vanish on $\partial\Omega_1$. Here and below a finite element function and its vector representation using the standard basic functions are denoted by the same letter for simplicity of notation. Note that the matrix form of (22) is as follows:

$$A_{II}^{(1)} H_1 v_\delta^{(1)} + A_{I\delta}^{(1)} v_\delta^{(1)} = 0 \tag{23}$$

Using the extension theorem, see for example Reference [12], we have

$$a_1(v_1, v_1) \leqslant C\rho_1 \|v_\delta^{(1)}\|^2_{H_{00}^{1/2}(\delta)} \tag{24}$$

Note that $B_\delta^{-1} B_\gamma v_\gamma^{(2)}$ is a vector representation of $\pi(v_2)$. Using (24), we have

$$(S_1 B_\delta^{-1} B_\gamma v_\gamma^{(2)}, B_\delta^{-1} B_\gamma v_\gamma^{(2)})_{R^{n_\delta}} = a_1(H_1\pi(v_2), H_1\pi(v_2)) \leqslant C\rho_1 \|\pi(v_2)\|^2_{H_{00}^{1/2}(\delta)}$$

It is known that the mortar projection is $H_{00}^{1/2}$-stable, see References [1, 2],

$$\|\pi(v_2)\|^2_{H_{00}^{1/2}(\delta)} \leqslant C\|v_2\|^2_{H_{00}^{1/2}(\gamma)} \tag{25}$$

Using this and the trace theorem, see for example Reference [12], and that $\rho_1 \leqslant \rho_2$, we get

$$(S_1 B_\delta^{-1} B_\gamma v_\gamma^{(2)}, B_\delta^{-1} B_\gamma v_\gamma^{(2)})_{R^{n_\delta}} \leqslant C a_2(H_2 v_\gamma^{(2)}, H_2 v_\gamma^{(2)}) \tag{26}$$

where $H_2 v_\gamma^{(2)}$ is the discrete harmonic extension of $v_\gamma^{(2)}$ given on γ and zero on $\partial\Omega_2\backslash\gamma$ in the sense of $(\nabla\cdot,\nabla\cdot)_{L^2(\Omega_2)}$. For $H_2 v_\gamma^{(2)}$ we have, see (21),

$$(S_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} = a_2(H_2 v_\gamma^{(2)}, H_2 v_\gamma^{(2)})$$

Using this in (26), we get (20). □

Remark
The preconditioner S_2 for S can be replaced by a matrix representation of the norm $H_{00}^{1/2}(\gamma)$. It is known that it is spectrally equivalent to S_2 and that in the case of a uniform mesh h_2 on γ FFT can be used for solving a system with this matrix, for details see Reference [13].

4.2. Neumann–Neumann preconditioner

We now define the N–N preconditioner for S. It is an extension of the N–N preconditioner known for the discretization of elliptic problems with discontinuous coefficients on matching triangulation, see References [4, 8, 12, 14], on non-matching triangulation. It is of the form

$$S_N = \left(\frac{2\rho_1}{\rho_1+\rho_2} B_\gamma^{\mathrm{T}} B_\delta^{-1} S_1^{-1} B_\delta^{-1} B_\gamma + \frac{2\rho_2}{\rho_1+\rho_2} S_2^{-1} \right)^{-1} \tag{27}$$

Theorem 4.2
Let h_γ/h_δ be uniformly bounded. For $v_\gamma^{(2)} \in R^{n_\gamma}$ and $\rho_1 \leqslant \rho_2$ holds

$$(S_N v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant (S v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant C (S_N v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{28}$$

where C is a positive constant independent of h_i and the coefficients ρ_i, $i=1,2$.

We first prove the following auxiliary result.

Lemma 4.2
Let h_γ/h_δ be uniformly bounded. For any $v_\gamma^{(2)} \in R^{n_\gamma}$ holds

$$(\hat{S}_1^{-1} B_\delta^{-1} B_\gamma v_\gamma^{(2)}, B_\delta^{-1} B_\gamma v_\gamma^{(2)})_{R^{n_\delta}} \leqslant C (\hat{S}_2^{-1} v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{29}$$

where $\hat{S}_i = S_i$ for $\rho_i = 1$, $i=1,2$, C is a positive constant independent of h_i and the coefficients ρ_i, $i=1,2$.

Proof
Note that, see for example Reference [15],

$$h_\delta^2 \|\hat{S}_1^{-1/2} B_\delta^{-1} B_\gamma v_\gamma^{(2)}\|_{R^{n_\delta}}^2 \leqslant C \|\pi(v_\gamma^{(2)})\|_{H^{-1/2}(\delta)}^2$$

where $\pi(v_\gamma^{(2)})$ is a function representation of $B_\delta^{-1} B_\gamma v_\gamma^{(2)}$, see (3) and (7), and $H^{-1/2}(\delta)$ is the dual space to $H_{00}^{1/2}(\delta)$. It can be proved, see the proof of Lemma 1 in Reference [16], that

$$\|\pi(v_\gamma^{(2)})\|_{H^{-1/2}(\delta)}^2 \leqslant C \left(1 + \frac{h_\delta}{h_\gamma}\right) \|v_\gamma^{(2)}\|_{H^{-1/2}(\gamma)}^2$$

Using this we have

$$\|\hat{S}_1^{-1/2} B_\delta^{-1} B_\gamma v_\gamma^{(2)}\|_{R^{n_\gamma}}^2 \leqslant C (\hat{S}_2^{-1} v_\gamma^{(2)}, v_\gamma^{(2)})_{R_\gamma^{(2)}}$$

since h_γ/h_δ is uniformly bounded by the assumption and the norm of $H^{-1/2}(\gamma)$ is equivalent to the norm generated by $h_\gamma^2 \hat{S}_2^{-1}$. □

Proof of Theorem 4.2
This reduces to proving that

$$(S^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant (S_N^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant C(S^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{30}$$

We have, see (27),

$$(S_N^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \geqslant \frac{2}{\rho_1+\rho_2}(\hat{S}_2^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{31}$$

Note that, see (18),

$$(\hat{S}_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} = \frac{1}{\rho_2}(S_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant \frac{1}{\rho_2}(S v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}}$$

Hence

$$(S^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant \frac{1}{\rho_2}(\hat{S}_2^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}}$$

Using this and $\rho_1 \leqslant \rho_2$ in (31) we get

$$(S_N^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \geqslant (S^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}}$$

which proves the left-hand side of (30).
By Lemma 4.2

$$(S_N^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant \frac{C}{\rho_1+\rho_2}(\hat{S}_2^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \tag{32}$$

By Lemma 4.1

$$(S v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant C(\rho_1+\rho_2)(\hat{S}_2 v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}}$$

Hence

$$\frac{1}{\rho_1+\rho_2}(\hat{S}_2^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}} \leqslant C(S^{-1}v_\gamma^{(2)}, v_\gamma^{(2)})_{R^{n_\gamma}}$$

Using this in (32), we get the right-hand side of (30). □

5. PRECONDITIONERS FOR (5)

In this section, we define and analyse preconditioners for problem (5). They will be defined, as in Section 4, for Schur complement systems with respect to unknowns λ_δ corresponding to λ^*, the Lagrange multipliers.

We consider system (8) rewritten in the form given by (10). We first eliminate the unknowns $u_I^{(1)}$ and $u_I^{(2)}$. Using rows 1 and 3 of (10) and substituting the result in rows 2 and 4 of (10) we obtain

$$\begin{pmatrix} S_1 & 0 & B_\delta \\ 0 & S_2 & -B_\gamma^{\mathrm{T}} \\ B_\delta & -B_\gamma & 0 \end{pmatrix} \begin{pmatrix} u_\delta^{(1)} \\ u_\gamma^{(2)} \\ \lambda_\delta \end{pmatrix} = \begin{pmatrix} F_\delta^{(1)} - (A_{I\delta}^{(1)})^{\mathrm{T}}(A_{II}^{(1)})^{-1}F_I^{(1)} \\ F_\gamma^{(2)} - (A_{I\gamma}^{(2)})^{\mathrm{T}}(A_{II}^{(2)})^{-1}F_I^{(2)} \\ 0 \end{pmatrix} \tag{33}$$

where S_1 and S_2 are given by (17).

Then, we eliminate the unknowns $u_\delta^{(1)}$ and $u_\gamma^{(2)}$ from this system. Using rows 1 and 2 of (33) we obtain

$$\tilde{S}_L\lambda_\delta = \tilde{F}_\lambda$$

where

$$\tilde{S}_L = B_\delta S_1^{-1} B_\delta + B_\gamma S_2^{-1} B_\gamma^{\mathrm{T}}$$

This system can be rewritten, pre-multiplying it by B_δ^{-1} and setting $\hat{\lambda}_\delta = B_\delta \lambda_\delta$, as

$$S_L\hat{\lambda}_\delta = F_\lambda \tag{34}$$

where

$$S_L = S_1^{-1} + B_\delta^{-1}B_\gamma S_2^{-1}B_\gamma^{\mathrm{T}}B_\delta^{-1} \tag{35}$$

and

$$F_\lambda = S_1^{-1}(F_\delta^{(1)} - (A_{I\delta}^{(1)})^{\mathrm{T}}(A_{II}^{(1)})^{-1}F_I^{(1)}) - B_\delta^{-1}B_\gamma S_2^{-1}(F_\gamma^{(2)} - (A_{I\gamma}^{(2)})^{\mathrm{T}}(A_{II}^{(2)})^{-1}F_I^{(2)})$$

The dual Schur complement matrix S_L is symmetric and positive definite, n_δ by n_δ.

Our goal is to define preconditioners for (34) which can be called as dual to those discussed in Section 4, i.e. dual to the Neumann–Dirichlet one and dual to the Neumann–Neumann one. The latter, for the matching triangulation, is called FETI, see References [6–8].

5.1. Dual Neumann–Dirichlet preconditioner

The Neumann–Dirichlet dual preconditioner for S_L is defined by S_1^{-1}.

Theorem 5.1

For any $\lambda \in R^{n_\delta}$ and $\rho_1 \leqslant \rho_2$ the following holds

$$(S_1^{-1}\lambda, \lambda)_{R^{n_\delta}} \leqslant (S_L\lambda, \lambda)_{R^{n_\delta}} \leqslant C(S_1^{-1}\lambda, \lambda)_{R^{n_\delta}} \tag{36}$$

where C is a positive constant independent of h_i and the coefficients ρ_i, $i = 1, 2$.

Proof

The left-hand side of (36) is obvious. A proof of the right-hand side follows from Lemma 5.1. □

Lemma 5.1
For any $\lambda \in R^{n_\delta}$ and $\rho_1 \leqslant \rho_2$ holds

$$(S_2^{-1} B_\gamma^{\mathrm{T}} B_\delta^{-1} \lambda, B_\gamma^{\mathrm{T}} B_\delta^{-1} \lambda)_{R^{n_\gamma}} \leqslant C (S_1^{-1} \lambda, \lambda)_{R^{n_\delta}} \tag{37}$$

where C is a positive constant independent of h_i and the coefficients ρ_i, $i = 1, 2$.

Proof
Setting $S_1^{-1/2} \lambda = t$ and then $w = B_\gamma^{\mathrm{T}} B_\delta^{-1} \lambda$, the proof reduces to showing that

$$\|S_2^{-1/2} w\|_{R^{n_\gamma}}^2 \leqslant C \|t\|_{R^{n_\delta}}^2 \tag{38}$$

We have

$$\|S_2^{-1/2} w\|_{R^{n_\gamma}}^2 = \sup_z \frac{|(S_2^{-1/2} w, z)_{R^{n_\gamma}}|^2}{\|z\|_{R^{n_\gamma}}^2} = \sup_v \frac{|(w, v)_{R^{n_\gamma}}|^2}{\|S_2^{1/2} v\|_{R^{n_\gamma}}^2}$$

Using that $w = B_\gamma^{\mathrm{T}} B_\delta^{-1} \lambda$ and the triangle inequality, we get

$$\begin{aligned} \|S_2^{-1/2} w\|_{R^{n_\gamma}}^2 &= \sup_v \frac{|(S_1^{-1/2} \lambda, S_1^{1/2} B_\delta^{-1} B_\gamma v)_{R^{n_\delta}}|^2}{\|S_2^{1/2} v\|_{R^{n_\gamma}}^2} \\ &\leqslant \sup_v \frac{\|t\|_{R^{n_\delta}}^2 \|S_1^{1/2} B_\delta^{-1} B_\gamma v\|_{R^{n_\delta}}^2}{\|S_2^{1/2} v\|_{R^{n_\gamma}}^2} \end{aligned}$$

Applying now Lemma 4.1, we get (38). □

5.2. FETI preconditioner

We now discuss FETI method for solving (34), dual to the Neumann–Neumann method which has been discussed in Section 4. This preconditioner is of the form

$$G = \left(\frac{\rho_2}{\rho_1 + \rho_2} S_1 + \frac{\rho_1}{\rho_1 + \rho_2} B_\delta^{-1} B_\gamma S_2 B_\gamma^{\mathrm{T}} B_\delta^{-1} \right)^{-1} \tag{39}$$

Theorem 5.2
Let h_γ / h_δ be uniformly bounded. For any $\lambda \in R^{n_\delta}$ and $\rho_1 \leqslant \rho_2$ holds

$$\tfrac{1}{2} (G\lambda, \lambda)_{R^{n_\delta}} \leqslant (S_L \lambda, \lambda)_{R^{n_\delta}} \leqslant C (G\lambda, \lambda)_{R^{n_\delta}} \tag{40}$$

where C is a positive constant independent of h_i and the coefficients ρ_i, $i = 1, 2$.

Proof
The left-hand side of (40) reduces to showing that

$$\frac{1}{2} (S_L^{-1} \lambda, \lambda)_{R^{n_\delta}} \leqslant (G^{-1} \lambda, \lambda)_{R^{n_\delta}} \tag{41}$$

Note that, see (35),

$$(S_L^{-1} \lambda, \lambda)_{R^{n_\delta}} \leqslant (S_1 \lambda, \lambda)_{R^{n_\delta}}$$

Using this and $\rho_1 \leqslant \rho_2$, we get, see (39),

$$(G^{-1}\lambda,\lambda)_{R^{n_\delta}} \geqslant \frac{\rho_2}{\rho_1+\rho_2}(S_1\lambda,\lambda)_{R^{n_\delta}} \geqslant \frac{1}{2}(S_L^{-1}\lambda,\lambda)_{R^{n_\delta}}$$

which proves (41).

By Lemma 5.1 we have

$$(S_L\lambda,\lambda)_{R^{n_\delta}} \leqslant C(S_1^{-1}\lambda,\lambda)_{R^{n_\delta}} \tag{42}$$

By Lemma 5.2, see below,

$$(G^{-1}\lambda,\lambda)_{R^{n_\delta}} \leqslant C\frac{\rho_2}{\rho_1+\rho_2}(S_1\lambda,\lambda)_{R^{n_\delta}}$$

Using this in (42) we get

$$(S_L\lambda,\lambda)_{R^{n_\delta}} \leqslant C\frac{\rho_2}{\rho_1+\rho_2}(G\lambda,\lambda)_{R^{n_\delta}} \leqslant C(G\lambda,\lambda)_{R^{n_\delta}}$$

which proves the right-hand side of (40). □

Remark

The preconditioner G can be replaced by

$$G = (\rho_1\hat{S}_1 + \rho_1 B_\delta^{-1}B_\gamma\hat{S}_2B_\gamma^{\mathrm{T}}B_\delta^{-1})^{-1}$$

Theorem 5.2 is also valid for this preconditioner with constant 1 instead of $\frac{1}{2}$ in the left-hand inequality of (40).

Lemma 5.2

Let h_γ/h_δ be uniformly bounded. For any $\lambda\in R^{n_\delta}$ holds

$$(\hat{S}_2B_\gamma^{\mathrm{T}}B_\delta^{-1}\lambda, B_\gamma^{\mathrm{T}}B_\delta^{-1}\lambda)_{R^{n_\gamma}} \leqslant C(\hat{S}_1\lambda,\lambda)_{R^{n_\delta}} \tag{43}$$

where $\hat{S}_i = S_i$ for $\rho_i = 1$, $i=1,2$, and C is a positive constant independent of h_i.

Proof

We have

$$\begin{aligned}\|\hat{S}_2^{1/2}B_\gamma^{\mathrm{T}}B_\delta^{-1}\lambda\|^2_{R^{n_\gamma}} &= \sup_z \frac{|(\hat{S}_2^{1/2}B_\gamma^{\mathrm{T}}B_\delta^{-1}\lambda, z)_{R^{n_\gamma}}|^2}{\|z\|^2_{R^{n_\gamma}}}\\ &= \sup_t \frac{|(\lambda, B_\delta^{-1}B_\gamma t)_{R^{n_\delta}}|^2}{\|\hat{S}_2^{-1/2}t\|^2_{R^{n_\gamma}}}\\ &\leqslant \sup_t \frac{\|\hat{S}_1^{1/2}\lambda\|^2_{R^{n_\delta}}\|\hat{S}_1^{-1/2}B_\delta^{-1}B_\gamma t\|^2_{R^{n_\delta}}}{\|S_2^{-1/2}t\|^2_{R^{n_\gamma}}}\end{aligned}$$

Using Lemma 4.2, we get

$$\|\hat{S}_2^{1/2}B_\gamma^{\mathrm{T}}B_\delta^{-1}\lambda\|^2_{R^{n_\gamma}} \leqslant C\|\hat{S}_1^{1/2}\lambda\|^2_{R^{n_\delta}}$$

which proves (43). □

6. IMPLEMENTATION ASPECTS

In this section, we discuss some implementation aspects of solving the Schur complement systems.

6.1. Implementation for (4)

To solve the Schur complement equation (14) we use the preconditioned conjugate gradient (PCG) iterations. Here, we only need to describe the implementation of two steps: (1) the multiplication of the Schur complement matrix $S \in R^{n_\gamma \times n_\gamma}$ (defined by (18)) by a given vector, and (2) solving a system with (a) the Neumann–Dirichlet preconditioner S_2 (defined by (17)), and with (b) the Neumann–Neumann preconditioner S_N (defined by (27)).

Let us recall that the iterations are carried out on the mortar side of the interface Γ with the number of grid points equal to n_γ. The mortar condition (3) ensures the proper transfer of information across the interface.

(1) Compute $r^k = Sw^k$ for any given $w^k \in R^{n_\gamma}$. The multiplication of S by a given vector reduces to solving two independent problems, see (18):

(i) Compute, see (17),

$$r_1^k = B_\gamma^{\mathrm{T}} B_\delta^{-1} S_1 B_\delta^{-1} B_\gamma w^k$$

We first compute $z_\delta^{(1)} = B_\delta^{-1} B_\gamma w^k$ solving the system $B_\delta z_\delta^{(1)} = B_\gamma w^k$ (this corresponds to the mortar condition (3)). Next we compute

$$S_1 z_\delta^{(1)} = A_{\delta\delta}^{(1)} z_\delta^{(1)} - A_{\delta I}^{(1)} (A_{II}^{(1)})^{-1} A_{I\delta}^{(1)} z_\delta^{(1)}$$

This reduces to solving the Dirichlet problem in Ω_1 as follows:

$$A_{II}^{(1)} v_I^{(1)} = A_{I\delta}^{(1)} z_\delta^{(1)} \tag{44}$$

and to computing $v_1 = A_{\delta\delta}^{(1)} z_\delta^{(1)} - A_{\delta I}^{(1)} v_I^{(1)}$.

Having v_1 we compute $r_1^k = B_\gamma^{\mathrm{T}} B_\delta^{-1} v_1$ solving the system in a similar way as above.

(ii) Compute, see (17),

$$r_2^k = S_2 w^k = A_{\gamma\gamma}^{(2)} w^k - A_{\gamma I}^{(2)} (A_{II}^{(2)})^{-1} A_{I\gamma} w^k$$

which reduces to solving the Dirichlet problem in Ω_2 as follows:

$$A_{II}^{(2)} v_I^{(2)} = A_{I\gamma}^{(2)} w^k \tag{45}$$

and to computing $r_2^k = A_{\gamma\gamma}^{(2)} w^k - A_{\gamma I}^{(2)} v_I^{(2)}$.

Finally, $r^k = r_1^k + r_2^k$.

(2a) Compute $d_\gamma^{(2)} = S_2^{-1} r^k$ for any given $r^k \in R^{n_\gamma}$, i.e. solve $S_2 d_\gamma^{(2)} = r^k$, which reduces, see (17), to solving the problem in Ω_2 of the form

$$\begin{pmatrix} A_{II}^{(2)} & A_{I\gamma}^{(2)} \\ A_{\gamma I}^{(2)} & A_{\gamma\gamma}^{(2)} \end{pmatrix} \begin{pmatrix} d_I^{(2)} \\ d_\gamma^{(2)} \end{pmatrix} = \begin{pmatrix} 0 \\ r^k \end{pmatrix} \tag{46}$$

This is the Neumann problem in Ω_2, to be exact, a problem with the non-homogeneous Neumann boundary condition on γ and the homogeneous Dirichlet one on $\partial\Omega_2\backslash\gamma$.

(2b) Compute $d_\gamma = S_N^{-1} r^k$ for any given $r^k \in R^{n_\gamma}$, i.e. solve $S_N d_\gamma = r^k$, which reduces to solving two independent problems, see (27):

(i) Compute $d_\delta^{(1)} = B_\gamma^{\mathrm{T}} B_\delta^{-1} S_1^{-1} B_\delta^{-1} B_\gamma r^k$. We first compute $g_\delta^{(1)} = B_\delta^{-1} B_\gamma r^k$. Then we compute $S_1^{-1} g_\delta^{(1)}$ which reduces to solving the Neumann problem on Ω_1, and more precisely, the problem with non-homogeneous Neumann boundary conditions on δ and homogeneous Dirichlet ones on $\partial\Omega_1\backslash\delta$. This problem is of the form, see (17), $S_1 d_\delta^{(1)} = g_\delta^{(1)}$ which reduces to solving

$$\begin{pmatrix} A_{II}^{(1)} & A_{I\delta}^{(1)} \\ A_{\delta I}^{(1)} & A_{\delta\delta}^{(1)} \end{pmatrix} \begin{pmatrix} d_I^{(1)} \\ d_\delta^{(1)} \end{pmatrix} = \begin{pmatrix} 0 \\ g_\delta^{(1)} \end{pmatrix} \tag{47}$$

Having $d_\delta^{(1)}$ we compute $d_\gamma^{(1)} = B_\gamma^{\mathrm{T}} B_\delta^{-1} d_\delta^{(1)}$.

(ii) Compute $d_\gamma^{(2)} = S_2^{-1} r^k$, which reduces to solving the problem on Ω_2 of (46).
Finally, $d_\gamma = (2\rho_1/(\rho_1+\rho_2))d_\gamma^{(1)} + (2\rho_2/(\rho_1+\rho_2))d_\gamma^{(2)}$.

6.2. *Implementation for* (5)

To solve the dual Schur complement equation (34) we use the preconditioned conjugate gradient (PCG) iterations. Here, we only need to describe the implementation of (1) the multiplication of the dual Schur complement matrix $S_L \in R^{n_\delta \times n_\delta}$ (defined by (35)) by a given vector, and (2) solving a system with (a) the Neumann–Dirichlet dual preconditioner S_1^{-1} (defined by (17)), and with (b) the Neumann–Neumann dual preconditioner G (defined by (39)).

Let us recall that the iterations are carried out on the non-mortar side of the interface Γ with the number of grid equal to n_δ. As before the mortar condition (3) ensures the proper transfer of information across the interface.

(1) Compute $r^k = S_L \lambda^k$ for any given $\lambda^k \in R^{n_\delta}$. The multiplication of S_L by a given vector reduces to solving two independent problems:

(i) Compute $r_1^k = S_1^{-1}\lambda^k$, i.e. solve

$$S_1 r_1^k = \lambda^k \tag{48}$$

This is the Neumann problem in Ω_1 as in (47) (see step (2b) described in Section 6.1).

(ii) Compute $r_2^k = B_\delta^{-1} B_\gamma S_2^{-1} B_\gamma^{\mathrm{T}} B_\delta^{-1} \lambda^k$. This step is similar to (46). The only difference is that before solving the Neumann problem on Ω_2 we need first to solve $B_\delta d_\delta^{(1)} = \lambda^k$, then to compute $B_\gamma^{\mathrm{T}} d_\delta^{(1)}$, and after solving the Neumann problem on Ω_2 we need to perform this transfer in reversed order.
Finally, $r^k = r_1^k + r_2^k$.

(2a) Compute $r_1^{(k)} = S_1 \lambda^k$ for any given $\lambda^k \in R^{n_\delta}$.
This step is the Dirichlet problem in Ω_1 similar to the step (1i) described in Section 6.1 (without the pre- and post-multiplications by $B_\delta^{-1} B_\gamma$ and its transpose, respectively).

(2b) Compute $r^{(k)} = G^{-1}\lambda^k$ for any given $\lambda^k \in R^{n_\delta}$. This step consists of solving two Dirichlet problems, one in Ω_1, the other in Ω_2 with properly scaled right-hand sides, see (39).

7. NUMERICAL EXPERIMENTS

The test example for all our experiments is the weak formulation, see (1), of

$$-\operatorname{div}(\rho(x)\nabla u) = f(x_1,x_2) \quad \text{in } \Omega \tag{49}$$

with the Dirichlet boundary conditions on $\partial\Omega$, where Ω is a union of two disjoint rectangular subregions Ω_i, $i=1,2$, of a diameter one, and $\rho(x)=\rho_i$ is a positive constant in each Ω_i.

Problem (49) is discretized by the finite element method on non-matching triangulation across the interface, Γ. The grids used in this study are: (1) *double grids*, where the grid on one side of the interface Γ is twice the one on the other side of Γ, with every other position of the nodes coinciding, (2) *staggered grids*, where the grid size, h on both sides of Γ is the same but the nodes are staggered, with the distance of $h/2$ between the nearest two nodes on the opposite sides of Γ, and (3) *mixed grids*, where the grid on one side of Γ is coarse with the grid size $2h$, while the grid on the other side of Γ is fine with the grid size h and staggered by $h/2$. The mixed grids may better represent general non-matching grids.

We select a face of Ω_2 and the interface Γ as the mortar side, while the face of Ω_1 is the non-mortar one. We choose the following combinations of the diffusion coefficients: (1) $\rho_1=\rho_2$, (2) $1=\rho_1<\rho_2=1000$, and (3) $1=\rho_2<\rho_1=1000$ (the case not covered by the theory).

To create a discrete driving function $f(x_1,x_2)$ we generate a random discrete solution $u(x_1,x_2)$ and multiply it by the matrix forms (12) and (10), for formulations (4) and (5), respectively.

We solve the problems using the preconditioned conjugate gradient (PCG) iterations (see Section 6 for the implementation aspects). The iterations are terminated when the norm of the residual has decreased 10^6 times in the norm generated by the inverse of the preconditioner. To estimate the condition number of the PCG iteration matrix, we compute the tridiagonal matrix representing the restriction of the preconditioned Schur complement matrix to the space spanned by the conjugate gradient residuals.

All four preconditioners considered behave as predicted by the theory: for $\rho_1 \leqslant \rho_2$ the convergence is independent of the grid size, see Table III and IV. Tables I and II present performance of all the preconditioners for the finest meshes on different grids. For discontinuous problems, when $\rho_1<\rho_2$ all four preconditioners (and especially the N–D and dual N–D ones) even display faster convergence than for the continuous problems; moreover, the N–D and dual N–D preconditioners converge somewhat faster than the N–N and dual N–N (FETI) preconditioners. When $\rho_2<\rho_1$ (the case not covered by the theory) all four preconditioners show somewhat erratic performance, see Tables I and II.

The differences in performance on different grids are qualitatively insignificant, thus in Tables III and IV we present only one set of experiments. Comparison of the convergence rate for the preconditioned and non-preconditioned iterations (on a chosen set of problems, see Tables III and IV) shows that the first remain constant independently of the grid size,

Table I. Performance of the Neumann–Dirichlet (N–D, $Q = S_2^{-1}S$) and (N–N, $Q = S_N^{-1}S$) preconditioners for the finest meshes on different grids.

Grids	Preconditioner	n_δ	n_γ	Continuous		$\rho_2 < \rho_1$		$\rho_1 < \rho_2$	
				No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$
Double	N–D	255	127	5	1.23	7	1.62	2	1.000
		127	255	7	2.85	12	*	2	1.002
	N–N	255	127	9	2.32	11	3.05	8	1.88
		127	255	10	4.37	20	*	6	1.61
Staggered	N–D	256	255	8	1.97	107	812	2	1.001
		255	256	10	3.41	115	*	3	1.30
	N–N	256	255	13	4.37	129	1036	9	2.70
		255	256	14	5.22	145	*	9	2.03
Mixed	N–D	256	127	6	1.32	8	1.89	2	1.000
		127	256	13	12.34	19	*	4	1.30
	N–N	256	127	11	5.07	14	7.19	10	3.84
		127	256	21	27.77	34	*	10	3.32

Note: The number of iterations and the estimate of the condition number are displayed. (*Denotes cases when the estimate fails).

Table II. Performance of the dual Neumann–Dirichlet (dual N–D, $Q = S_1 S_L$) and dual Neumann–Neumann (dual N–N, or FETI, $Q = G^{-1}S_L$) preconditioners for the finest meshes on different grids.

Grids	Preconditioner	n_δ	n_γ	Continuous		$\rho_2 < \rho_1$		$\rho_1 < \rho_2$	
				No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$
Double	Dual N–D	255	127	5	2.00	10	*	2	1.001
		127	255	4	1.34	6	1.85	2	1.001
	Dual N–N	255	127	11	9.97	23	*	7	5.00
		127	255	6	1.73	7	2.26	5	1.28
Staggered	Dual N–D	256	255	8	1.93	115	997	3	1.30
		255	256	9	3.08	114	1176	2	1.002
	Dual N–N	256	255	13	4.27	144	1003	9	2.85
		255	256	12	5.07	146	2957	8	1.91
Mixed	Dual N–D	256	127	7	2.28	16	*	3	1.31
		127	256	10	10.98	13	91.0	3	1.01
	Dual N–N	256	127	14	19.23	35	*	12	9.98
		127	256	15	22.21	18	181.7	8	2.96

Note: The number of iterations and the estimate of the condition number are displayed. (*Denotes cases when the estimate fails).

while the latter depend roughly proportional to the square root of the size of the iteration matrix†, n_γ for formulation (4) and n_δ for formulation (5).

†The number of iterations for the non-preconditioned problems in the range $n = 16$ to 256 ($n = n_\gamma$ for Table III and $n = n_\delta$ for Table IV) is proportional to n^p, where $p = 0.5 \pm 0.03$ as computed using *polyfit* in the log–log scale.

Table III. Examples of the PCG iterations convergence for (4) with and without preconditioners (for $\rho_1 < \rho_2$) as a function of grid sizes on the mixed grids.

n_δ	n_γ	No precond.		N–D precond.		N–N precond.	
		No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$
16	7	7	6.31	2	1.00	7	3.53
32	15	15	13.06	2	1.00	10	3.80
64	31	24	26.34	2	1.00	10	3.83
128	63	33	52.76	2	1.00	10	3.85
256	127	47	105.62	2	1.00	10	3.84
7	16	16	14.54	4	1.30	8	3.12
15	32	24	27.91	4	1.30	10	3.29
31	64	34	54.29	4	1.30	10	3.31
63	128	44	107.0	4	1.30	10	3.32
127	256	62	212.3	4	1.30	10	3.32

Note: The number of iterations and estimate of the condition number are displayed.

Table IV. Examples of the PCG iterations convergence for (5) with and without preconditioners (for $\rho_1 < \rho_2$) as a function of grid sizes on the mixed grids.

n_δ	n_γ	No precond.		Dual N–D precond.		Dual N–N precond.	
		No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$
16	7	12	14.35	4	1.30	9	9.88
32	15	18	27.06	4	1.30	12	9.96
64	31	24	52.39	4	1.31	12	9.97
128	63	33	102.9	3	1.31	12	9.98
256	127	44	203.8	3	1.31	12	9.98
7	16	7	6.29	3	1.01	7	2.81
15	32	11	12.95	3	1.01	8	2.96
31	64	17	25.68	3	1.01	8	2.96
63	128	23	51.29	3	1.01	8	2.96
127	256	33	102.7	3	1.01	8	2.96

Note: The number of iterations and estimate of the condition number are displayed.

Additionally, we performed experiments with the grid ratio across the interface varying in the range $h_\gamma/h_\delta = 2^k, k = -5(1)5$, i.e. from $\frac{1}{32}$ to $\frac{32}{1}$ (and different diffusion coefficients ρ, as before). Performance for the dual N–D preconditioner was virtually independent of the grid ratio, as was for the FETI preconditioner and $h_\gamma/h_\delta < 1$. For $h_\gamma/h_\delta > 1$ the condition number of the FETI iteration matrix grows almost quadratically with the grid ratio while the number of iterations increases only very slowly, see Table V (it also depends on the grid size).

8. CONCLUSIONS

All four preconditioners considered behave as predicted by the theory: for $\rho_1 \leqslant \rho_2$ the convergence is independent of the grid size and the jump of the discontinuity. The general

Table V. The PCG iterations convergence with the FETI preconditioner for different grid ratios $h_\gamma/h_\delta > 1$ for $\rho_1 = \rho_2$.

n_γ	$h_\gamma/h_\delta = 2:1$		$h_\gamma/h_\delta = 4:1$		$h_\gamma/h_\delta = 8:1$		$h_\gamma/h_\delta = 16:1$		$h_\gamma/h_\delta = 32:1$	
	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$	No. it.	$\kappa(Q)$
7	9	9.7	9	33.1	9	126.1	9	498.5	12	1951
15	12	9.9	16	33.7	17	129.0	17	510.0	—	—
31	12	10.0	16	33.9	19	129.7	—	—	—	—
63	11	10.0	15	33.9	—	—	—	—	—	—

Note: The number of iterations and estimate of the condition number are displayed.

observation is that all preconditioners considered are very robust for cases with the discontinuity ratio of 1000 across the interface. One should be cautious not to generalize conclusions drawn on such limited two subdomain case. Nevertheless, the results are illuminating, and we intend to extend the experimental evidence to more complex subdivisions.

The discussed preconditioners can be used for the mortar discretization with the mortar space defined by the dual basis functions, see Reference [5]. The analysis of convergence presented in this paper is also valid for that discretization. In view of experiments reported in Reference [5], we expect numerical results similar to the ones reported in this paper.

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