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NUMERICAL CALCULATIONS OF THE INFLUENCE OF LARGE EARTH CURRENTS ON A BURIED CABLE

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Abstract—In certain applications one needs to determine the potential generated in the electrical cable by the earth current and leakage currents. In this paper a simple numerical method to solve this problem is considered. The results of the numerical experiments indicate that the proposed scheme is highly accurate.

INTRODUCTION

The aim of this research is to investigate the possibilities of numerically solving the following electrical engineering problem:

An uncovered cable is buried in the earth at a certain depth. It passes in the vincity of a source of current of a given magnitude. One needs to determine the potential generated in the cable by this earth current and by leakage currents from the cable. Additionally, the leakage and longitudinal currents in the cable should also be calculated.

1. FORMULATION OF THE PROBLEM

Let the network C be buried in the earth at a constant depth d. Each branch is a straight conductor with a constant radius a and resistance per unit length R. Some points in the network may be grounded. Grounds are assumed to be vertical rods (also straight conductors with constant radius and resistivity).

The potential $V(\mathbf{x})$ at the point $\mathbf{x} = (x, y, z)$ is generated by an earth current of given magnitude and by leakage currents from the buried cable including the grounds and is given by the equation

$$V(\mathbf{x}) = V^0(\mathbf{x}) + \rho \int_{\mathbf{x}_c \in C} K(\mathbf{x}, \mathbf{x}_c) I(\mathbf{x}_c)\, d\mathbf{x}_c \tag{1}$$

Where

$V^0(\mathbf{x})$ is potential generated by earth current I_s:

$$V^0(\mathbf{x}) = I_s \cdot \frac{\rho}{2\pi} \cdot \frac{1}{|\mathbf{x} - \mathbf{x}_0|},$$

$\mathbf{x}_0 = (x_0, y_0, z_0)$ is the source of earth current,
$I(\mathbf{x}_c)$ is leakage current from the cable:

$$I(\mathbf{x}_c) = \frac{1}{R(\mathbf{x}_c)} \cdot \frac{\partial^2 V}{\partial \zeta^2}(\mathbf{x}_c).$$

Where

$(\partial/\partial\zeta)$ is the directional derivative along the cable,
$\mathbf{x}_c$ is point on the cable's axis,

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ρ is earth resistivity (Ω/m),
R is cable resistivity (Ω/m),
integral's kernel $K(\mathbf{x}, \mathbf{x}_c)$ is impedance between points $\mathbf{x}$ and $\mathbf{x}_c$.
In the case of a semi-infinite medium

$$K(\mathbf{x}, \mathbf{x}_c) = K_1(\mathbf{x}, \mathbf{x}_c) + K_2(\mathbf{x}, \mathbf{x}_c) = K_1(\mathbf{x}, \mathbf{x}_c) + K_1(\mathbf{x}, \hat{\mathbf{x}}_c),$$

where $\hat{\mathbf{x}}_c$ is reflection point of $\mathbf{x}_c$ above the earth's surface and K_1 is defined by

$$K_1(\mathbf{x}, \mathbf{x}_c) = \frac{1}{4\pi} \cdot \frac{1}{|\mathbf{x} - \mathbf{x}_c|}.$$

From physical considerations the following conditions must be satisfied:
(a) continuity of potential $V(\mathbf{x})$, $\mathbf{x} \in C$,
(b) Kirchhoff's law on longitudinal current $J(\mathbf{x})$, $x \in C$, where $J(\mathbf{x}) = -(1/R(\mathbf{x}))\cdot(\partial V/\partial \zeta)(\mathbf{x})$,
(c) $\int_{\mathbf{x} \in C} I(\mathbf{x})\, d\mathbf{x} = 0.$

2. DISCRETE MODEL, ASSUMPTIONS AND DEFINITIONS

We begin with a formal description of the networks in terms of tree structures later used for the formulation of the numerical scheme. Let C be a network in the horizontal plane (x, y) which has a tree structure. This means formally (Knuth[1]) that C is a finite set of one or more nodes such that:

(i) There is one specially designated node called the root of the tree, (ii) The remaining nodes are partitioned into $m > 0$ disjoint sets $C_1, \ldots, C_m$, and each of these sets in turn is a tree. This implies that C is a finite connected graph without cycles which contains n nodes and $n - 1$ branches.

The level of a node with respect to C is defined by saying that the root has level 1, and other nodes have a level that is one higher than they have with respect to the subtree of the root, C_j, which contains them. Call successors of node Q all nodes with level of one higher than Q such that they belong to subtrees of node Q. Node Q is then called the predecessor of its successors. Notice that predecessor is defined uniquely. Call a node terminal if it has no successors. Call all nodes which belong to subtree of node Q descendants of node Q, i.e. a successor is an immediate descendant. Similarly, a predecessor is an immediate ancestor. Denote by $\mathcal{D}(Q)$ the set of all descendants of Q.

Traverse the tree in preorder. This traversal proceeds according to the following steps:

(i) Visit the root. (ii) Traverse the left subtree (in preorder). (iii) Traverse the remaining subtrees (in preorder).

Attach number Q to the root of the tree. Consecutively number the nodes in accordance with traversal. Call a node maximal for node Q that descendant of Q which has the maximal number.

Attach to the branches the highest node number out of two nodes which the branches are connecting. Thus each branch has its beginning (B) and its end (E), where $B < E$.

From now on we shall consider trees defined as above, the nodes of which are the centers of cable segments. A ground is considered to be an ordinary branch attached to the nearest node with the highest level. Ground nodes are here always terminal ones. Attach the value $G = 1$ to grounds, the value $G = 0$ to all other branches.

3. NUMERICAL METHOD

Let the root of the tree be the origin of an orthogonal system of coordinates (x, y, z) with one axis (z) perpendicular to the earth's surface. Let us divide the network into meshes, the step size h being constant within a branch. For each step we introduce a local Cartesian system of coordinates (ξ, η, σ) with origin at the midpoint of the step and with abscissa ξ along the cable segment.

Let us assume further than the $I(\xi)$ is piecewise constant:

$$I(\xi) = I_l \text{ for } \xi \in [-h_l/2, h_l/2] \tag{2}$$

where h_l is the step size of the lth cable segment.

The assumptions on the leakage current $I(x)$ imply that the potential $V(x)$ is a piecewise second order polynomial in x:

$$V_k(x) = a_k x^2 + b_k x + c_k \quad x \in [-h_k/2, h_k/2].$$

The conditions mentioned in Section 1 are therefore:

(a) *Continuity of potential*

$$V_k(h_k/2) = V_p(-h_p/2),$$

for any node $k = 1, \ldots, n$ but the terminal ones and, for any p successors of k.

(b) *Conservation (Kirchhoff's) law*

$$J_k(h_k/2) = \sum J_p(-h_p/2) \quad p\text{-successors of } k,$$

for any nodes $k = 1, \ldots, n$ but the terminal ones.

(b′) *With boundary conditions*

$$J_1(-h_1/2) = J_m(h_m/2) = 0,$$

for any terminal nodes m (including groundings).

Where J is the current along the network C

$$-J(x) \cdot R(x) = V'(x) \text{ and } I(x) = -J'(x) = V''(x)/R(x).$$

If we denote every function $F_k(0)$ as F_k then

$$V_k = c_k$$
$$V_k' = -J_k R_k = b_k$$
$$V_k'' = I_k R_k = 2a_k$$

and thus

$$\left.\begin{aligned} V_k(x) &= I_k R_k x^2/2 - J_k R_k x + V_k \\ J_k(x) &= -I_k x + J_k \\ I_k(x) &= I_k \end{aligned}\right\} x \in [-h_k/2, h_k/2]. \tag{3}$$

From this assumption follows:

$$\int_{\mathbf{x}_c \in C} K(\mathbf{x}, \mathbf{x}_c) I(\mathbf{x}_c)\, \mathrm{d}\mathbf{x}_c = \sum_{l=1}^{n} \int_{-h_l/2}^{h_l/2} K(\xi, \xi_c) \cdot I(\xi_c)\, \mathrm{d}\xi_c = \sum_{l=1}^{n} \psi_l(\xi, h_l) \cdot I_l, \tag{4}$$

where

$$\psi_l(\xi, h_l) = \int_{-h_l/2}^{h_l/2} K(\xi, \xi_c)\, \mathrm{d}\xi_c = \frac{1}{4\pi} \int_{-h_l/2}^{h_l/2} \left(\frac{1}{|\xi - \xi_c|} + \frac{1}{|\xi - \hat{\xi}_c|} \right) \mathrm{d}\xi_c, \tag{5}$$

$\mathbf{x} = (x, y, z)$ is field point,
$\mathbf{x}_c = (x_c, y_c, z_c)$ is the midpoint of the lth cable segment,
$\boldsymbol{\xi} = (\xi, \eta, \sigma)$ and $\xi_c = (\xi_c, \eta_c, \sigma)$ are respectively, points $\mathbf{x}$ and $\mathbf{x}_c$ in the local system of coordinates of the lth step,
$\hat{\boldsymbol{\xi}}_c = (\hat{\xi}_c, \hat{\eta}_c, \hat{\sigma}_c)$ is reflection point of $\boldsymbol{\xi}_c$ above the earth's surface,
n is number of cable segments.

If point $\mathbf{x}$ (or for that matter $\boldsymbol{\xi}$) is also on the cable surface and is the kth center of the network then

$$\psi_l(\boldsymbol{\xi}^k, h_l) = \psi_{k,l}(h_l).$$

As

$$\int_{-A}^{A} \frac{\mathrm{d}u}{\sqrt{((a-u)^2+b^2)}} = \ln(a-u+\sqrt{(a-u)^2+b^2)})\Big]_{u=A}^{u=-A}$$

then

$$\psi_{k,l}(h_l) = \frac{1}{4\pi}\left\{\ln\frac{\xi+h_l/2+\sqrt{((\xi+h_l/2)^2+\eta^2+\sigma^2)}}{\xi-h_l/2+\sqrt{((\xi-h_l/2)^2+\eta^2+\sigma^2)}} + \ln\frac{\hat{\xi}+h_l/2+(\sqrt{((\hat{\xi}+h_l/2)^2+\hat{\eta}^2+\hat{\sigma}^2)}}{\hat{\xi}-h_l/2+\sqrt{((\hat{\xi}-h_l/2)^2+\hat{\eta}^2+\hat{\sigma}^2)}}\right\} \tag{6}$$

where $\sqrt{(\eta^2+\sigma^2)} \geq a_l$ is the radius of the lth segment.
Denote

$$r = \sqrt{(\xi^2+\eta^2+\sigma^2)} \text{ and } \hat{r} = \sqrt{(\hat{\xi}^2+\hat{\eta}^2+\hat{\sigma}^2)}.$$

Then for $r \gg h$ and $\hat{r} \gg h$ we have the following series expansion of eqn (6)

$$\psi_{k,l}(h_l) = \frac{1}{4\pi}\left\{\left(\frac{h}{r}+\frac{h}{\hat{r}}\right) + \frac{1}{24}\left[(3\alpha-1)\left(\frac{h}{r}\right)^3 + (3\hat{\alpha}-1)\left(\frac{h}{\hat{r}}\right)^3\right] + 0\left(\left(\frac{h}{r}\right)^5 + \left(\frac{h}{\hat{r}}\right)^5\right)\right\},$$

where

$$\alpha = \frac{\xi}{r} \text{ and } \hat{\alpha} = \frac{\hat{\xi}}{\hat{r}}.$$

The following special cases occur in our scheme:

1. Segment l is in the horizontal plane (x, y) at a distance d from the earth's surface and center of the k segment is at a distance z from the earth's surface and $z \geq d \geq 0$.

Then $\sigma = z - d$, $\hat{\sigma} = z + d$

$$\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\end{pmatrix} = \begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\cos\alpha & \sin\alpha\\ -\sin\alpha & \cos\alpha\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix},$$

where α is the angle between the integrated segment and abscissa, and

$$\mathbf{x} = (x, y) = (x^k - x^l, y^k - y^l).$$

2. The lth segment is vertical (i.e. a ground rod, with its center at a distance $h_l/2$ from the earth surface) and the center of the k segment is at a distance z from the earth's surface.

Then $\xi = h_l/2 + z$, $\hat{\xi} = -h_l/2 + z$,

$$\eta^2+\sigma^2 = r^2 = (x^k-x^l)^2 + (y^k-y^l)^2 \geq a^2,$$

$(\hat{\eta}, \hat{\sigma}) = (\eta, \sigma)$.

Where a is the radius of a grounding rod. In this case formula (6) can be simplified to

$$\psi_{k,l}(h_l) = \frac{1}{4\pi}\ln\frac{z+h_l+\sqrt{((z+h_l)^2+r^2)}}{z-h_l+\sqrt{((z-h_l)^2+r^2)}}.$$

The basic eqn (1) has the following descrete representation under the above—formulated assumptions:

$$V(\xi^k) = V^0(\xi^k) + \rho \sum_{l=1}^{n} \psi_l(\xi^k, h_l) \cdot I_l \tag{7}$$

or in simplified notation

$$V_k = V_k^0 + \rho \sum_{l=1}^{n} \psi_{k,l}(h_l) I_l, \text{ valid for any } k = 1, \ldots, n.$$

This set of n linear algebraic equations can be written in a compact matrix form as

$$\mathbf{V} = \mathbf{V}^0 + \rho\boldsymbol{\psi}\mathbf{I}, \tag{8}$$

where $\mathbf{V}, \mathbf{V}^0, \mathbf{I}$ are vectors and $\boldsymbol{\psi}$ is matrix $(n \times n)$.

Equation (8) gives us a system of n equations connecting the unknowns V and I. The remaining two sets of n equations can each be obtained from the conservation law and continuity of the potential. Using the conservation law and the boundary conditions, $J_m(h_m/2) = 0$, and working backwards from the terminal nodes towards the root, one can show by induction that the following holds for any $k = 1, \ldots, n$

$$J_k = I_k h_k/2 + \sum_{l \in \mathscr{D}(k)} I_l h_l. \tag{9}$$

The boundary condition at the root, $J_1(-h_1/2) = 0$, gives us one additional equation

$$\sum_{k=1}^{n} I_k h_k = 0, \tag{10}$$

which is a discretized form of the condition

$$\int_{\mathbf{x} \in C} I(x)\,\mathrm{d}\mathbf{x} = 0.$$

Similarily, continuity of the potential gives us, after some algebraic manipulation and using eqn (9) to eliminate J_k, the following system of $(n-1)$ equations, for any $l = 2, \ldots, n$ where node k is a predecessor of node l

$$V_k - V_l = \gamma_k\left(I_k h_k/4 + \sum_{i \in \mathscr{D}(k)} I_i h_i\right) + \gamma_l\left(\frac{3}{4} I_l h_l + \sum_{j \in \mathscr{D}(l)} I_j h_j\right). \tag{11}$$

where $r_l = h_l R_l/2$. Altogether formulas (8)–(11) give $3n$ linear equations for $3n$ unknowns V_k, J_k and I_k, $k = 1, \ldots, n$.

Formulas (11) and (10) in algebraic notation have the form:

$$\mathbf{CV} = \mathbf{DI} \text{ where } \mathbf{C} \text{ and } \mathbf{D} \text{ are } n \times n \text{ matrices.} \tag{12}$$

Introducing eqn (12) into (8) we obtain

$$(\mathbf{E} - \rho\boldsymbol{\psi}\mathbf{D}^{-1}\mathbf{C})\mathbf{V} = \mathbf{V}^0 \tag{13}$$

where $\mathbf{E}$ unit matrix, $n \times n$.

This is the final set of n algebraic equations to be solved. From its solution (potential V_k) other unknowns (I_k and J_k) can be easily calculated by means of eqns (12) and (9).

4. NUMERICAL RESULTS OF THE TEST RUNS

The test case was suggested by Televerket from practical considerations. The network is represented on Fig. 1. Through the source A flows current I_s of 1000 Amp. The uncovered cable

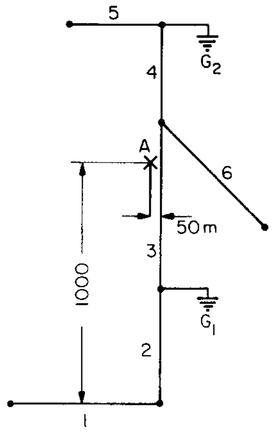

Fig. 1.

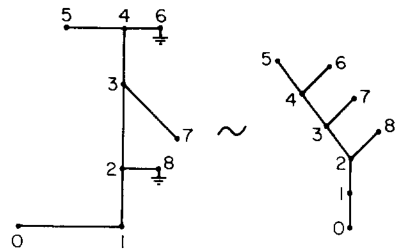

Fig. 2.

Fig. 1. The network.

Fig. 2. Correspondence between the network and the tree structure.

is buried at a constant depth of 0.6 m. Earth resistivity ρ is 3000 Ω/m. The rest of the input data is collected in Table 1. Note that originally for groundings $G1$ and $G2$ only resistances to earth were given, 10 and 75 Ω, respectively. The corresponding data in Table 1 was computed approximately using eqn (3.25) in Sunde[3]. Figure 2 illustrates the discrete model which was introduced in Section 2.

The tests were run on an IBM 360/75 computer. In the case of a simple homogeneous cable, the numerical results indicate (see Table 2 and Figs. 3–5), as expected, that the potential, leakage and longitudinal currents all are smooth functions. The leakage current grows very rapidly in the vicinity of the source and the conservation law for the longitudinal current is preserved in the triplet point of branches Nos. 3, 4 and 6 and in branchings to grounds. The results of experiments with different step sizes h suggest that the accuracy of the method is $O(h^3)$. This indicates the possibility of efficiently using some extrapolation schemes, if necessary.

All the eqns (8)–(11) were solved straightforwardly by using direct methods and no attempt was made to increase the efficiency of the method in that respect. The programs used are included in the original report[2].

Table 1. Input data

branch	cable length m	cable radius $\times 10^2$	cable resistivity $\times 10^3$
	m	m	ohm/m
1	600	0.785	3.05
2	500	2.45	0.62
3	700	3.11	0.43
4	400	2.45	0.62
5	400	1.07	1.95
6	600	1.07	1.95
G1	550	1.07	1.95
G2	60	1.07	1.95

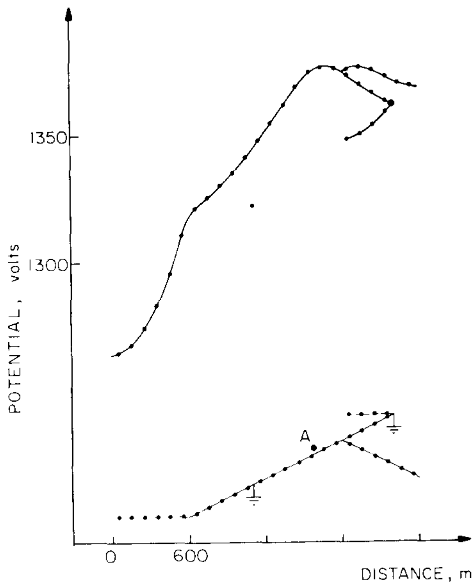

Fig. 3.

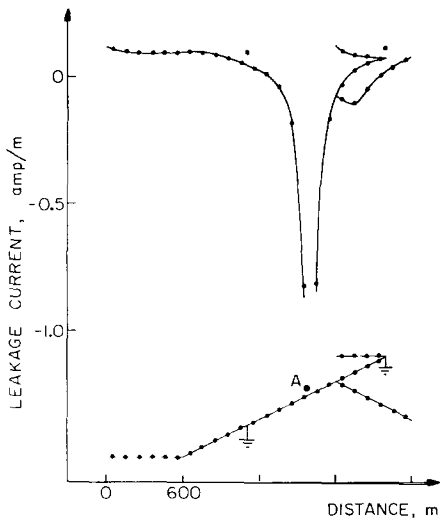

Fig. 4.

Fig. 3. Schematic representation of the horizontal network and the graph of the corresponding potential computed with the step size $h = 100$ m.

Fig. 4. Schematic representation of the horizontal network and the graph of the corresponding leakage current computed with the step size $h = 100$ m.

Table 2. Numerical results of the test run with the step $h = 100$ m (note though that grounds are represented by one mesh only). Each value is associated with the mid-point of the corresponding mesh. For the numbering of branches refer to Fig. 1

Mesh #	Branch #	Potential	Leakage current	Longitudinal current	Earth current potential
1	1	1264.32	0.110328	-5.5164	427.057
2	1	1267.64	0.100534	-16.0596	443.315
3	1	1274.06	0.096685	-25.9205	457.328
4	1	1283.43	0.094786	-35.4940	468.119
5	1	1295.70	0.094233	-44.9450	475.095
6	1	1310.85	0.094686	-54.3909	477.465
7	2	1321.42	0.101467	-64.1986	501.900
8	2	1325.71	0.094794	-74.0116	560.734
9	2	1330.58	0.086671	-83.0849	635.210
10	2	1335.60	0.075149	-91.1759	732.397
11	2	1341.86	0.054654	-97.6660	864.552
12	3	1348.30	0.037641	-157.3412	1054.542
13	3	1355.13	0.015573	-160.0002	1350.473
14	3	1362.01	-0.040530	-158.7540	1872.766
15	3	1368.76	-0.186912	-147.3820	3019.731
16	3	1374.27	-0.822959	-96.8885	6752.129
17	3	1376.67	-0.816802	-14.9004	6752.129
18	3	1375.90	-0.165191	34.1993	3019.731
19	4	1373.39	-0.024631	55.1162	1872.766
20	4	1369.94	0.029308	54.8823	1350.473
21	4	1366.65	0.058524	50.4907	1054.542
22	4	1363.71	0.072466	43.9412	864.552
23	5	1359.30	0.074478	30.0211	795.774
24	5	1354.18	0.080224	22.2830	784.947
25	5	1350.63	0.085537	13.9919	754.938
26	5	1348.77	0.097151	4.8576	711.762
27	6	1375.99	-0.084247	-7.2135	2574.551
28	6	1376.53	-0.103439	2.1708	2621.196
29	6	1375.23	-0.047725	9.7290	2094.479
30	6	1373.00	0.006612	11.7848	1584.923
31	6	1370.86	0.042948	9.3068	1234.700
32	6	1369.53	0.071594	3.5797	999.222
33	G1	1322.79	0.100110	27.5302	833.531
34	G2	1362.12	0.109548	3.2864	792.043

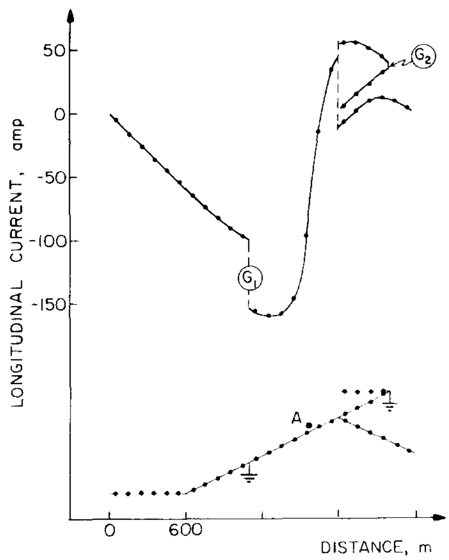

Fig. 5. Schematic representation of the horizontal network and the graph of the corresponding longitudinal current computed with the step size $h = 100$ m.

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REFERENCES

1. D. E. Knuth, *The Art of Computer Programming*, Vol. 2. Addison-Wesley, New York (1968).
2. W. Proskurowski and S. Linde, Report NA71.30, Institute of Technology, Stockholm.
3. E. D. Sunde, *Earth Conduction Effects in Transmission Systems*. Van Nostrand, New York (1949).