Project – Math 502b

The motion of two bodies under mutual gravitational attraction is described by the following equations derived from Newton's Law of Motion.

$$x''(t) = -\alpha^2 x(t)/R(t),$$
$$y''(t) = -\alpha^2 y(t)/R(t),$$
$$x(0) = 1 - \beta, \quad x'(0) = 0, \quad y(0) = 0, \quad y'(0) = \alpha\sqrt{\frac{1+\beta}{1-\beta}},$$

where x(t) and y(t) denote the position of one body in a coordinate system with the origin fixed in the other body, $R(t) = (x^2(t) + y^2(t))^{3/2}$, α and β are constants, $0 \leq \beta < 1$. The orbit is then an ellipse with eccentricity β and one focus at the origin. Choose $\alpha = \frac{\pi}{4}$ and β such that $1 - \beta = 10^{-i}$, $i = 0 : 5$, i.e. β approaching 1 as closely as possible.

1. Write a C–program based on the classical Runge-Kutta 4^{th}–order method. Test your code with a fixed step size h on the problem above. Verify the theoretical dependence of accuracy on the step size, h.
Note that the solution is periodic, with the period P = 8. As $(1 - \beta) \rightarrow 1$ the ellipse becomes more pronounced and the equations become *stiff*, i.e. more 'difficult' to integrate. Estimate the largest (in modulus) eigenvalue of the Jacobian as a function of $(1 - \beta)$ and find experimentally the largest step size for which the method converges.
2. Implement an automatic step size control. Norm of the global error at T = 8 should be close to a prescribed tolerance ϵ. Choose $\epsilon = 10^{-4}$ to 10^{-8} and estimate the norm of the local error, δ, at each grid point using two different step sizes. A computed value with the step size h can be accepted if the local error $\delta < \tau = \frac{(1-\beta)\epsilon h}{T}$. Otherwise the mesh must be refined. The increase and decrease of the step size should be regulated by $h_{new} = h_{old}\sqrt[p]{\frac{\theta\tau}{\delta}}$, where p is the order of the method, and θ is a safety factor in the range 0.8 to 0.9.
3. Use the set of all available Matlab solvers *odexxx* possibly changing 'options' from their default values. Note: you may determine the period using 'events'.
For the set of printout points $t_k = \frac{k}{2}, k = 0, ..., 16$ print out the position and *for all the steps taken* plot the position (x(t),y(t)), velocity (x'(t),y'(t)) and acceleration (x"(t),y"(t)). Note: you must force the computational points to coincide with the set of printout points even if a longer step is permitted by the estimator.

Plot (and print out) the number of steps taken (or better, the number of function evaluations) between the printout points t_k. Explain why these numbers vary in different regions of integration. Also print out the smallest step size h_{min} taken (and at which t). Observe how close the bodies come to each other.
Compare the efficiency and accuracy of the results for different values of the tolerance ϵ. Efficiency can be measured as the number of function calls or the cpu time. Error can be measured as the discrepancy in the positions (x(t),y(t)) for t and t+P for t=0.
Compare the efficiency of implementations in **1** to **3**.