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RATES OF CONVERGENCE OF NUMERICAL METHODS FOR CONTROLLED REGIME-SWITCHING DIFFUSIONS WITH STOPPING TIMES IN THE COSTS*

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Abstract. This work is concerned with rates of convergence of Markov chain approximation methods for controlled switching diffusions. The cost function is defined on an infinite horizon with stopping times and without discount. Displaying both continuous dynamics and discrete events, the discrete events are modeled by continuous-time Markov chains to delineate a random environment and other random factors that cannot be represented by diffusion processes. This paper presents a first attempt using a probabilistic approach to treat such rates of convergence problems. In addition, in contrast to the significant developments in the literature using partial differential equation (PDE) methods for the approximation of controlled diffusions, there do not yet appear to be any PDE results to date for rates of convergence of numerical solutions for controlled switching diffusions, to the best of our knowledge. Although some of the working conditions in this paper such as the one-dimensional continuous state variable, nondegenerate diffusions, and control only on the drift may be seemingly strong, they are adequate as the starting point for using this new approach to treat the rates of convergence problems. Moreover, in the literature, to prove the convergence using Markov chain approximation methods for control problems involving cost functions with stopping (even for uncontrolled diffusion without switching), an added assumption was used to avoid the so-called tangency problem. As a by-product of our approach, by modifying the value function, it is demonstrated that the anticipated tangency problem will not arise in the sense of convergence in probability and in the sense of L^1.

Key words. controlled switching diffusion, Markov chain approximation, rate of convergence, dynamic programming equation

AMS subject classifications. 93E20, 93E25, 60J60

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1. Introduction.

This work is concerned with rates of convergence of Markov chain approximation methods for controlled switching diffusions. The cost function is defined on an infinite horizon with stopping times and without discount. The regime switching is modeled by a continuous-time Markov chain. The motivation for using such models stems from the need to deal with emerging applications in manufacturing systems, financial engineering, and wireless communications; see [7, 10, 24, 25] and the references therein. The reader is referred to [26] for some recent developments on recurrence and ergodicity of switching diffusion processes; see also [18] for related stability problems. The added regime-switching component provides more flexibility to model the real-world scenario. Meanwhile, it causes much difficulty in the analysis and numerical treatment of the associated control and optimization problems.

Numerical methods using Markov chain approximation for controlled diffusions have been studied extensively in [15, 17], among others. A systematic approach for proving the convergence of the algorithms using probability methods was given in

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the aforementioned references, although the associated convergence rates were not discussed there. Owing to its importance, a great deal of attention has been devoted to the convergence rate problems; see [1, 2, 4, 5, 12, 13, 14, 19, 23] and the references therein. As was demonstrated in these references, rates of convergence may be treated by considering convergence rates of the finite difference scheme for Hamilton–Jacobi–Bellman (HJB) equations. Most of the work considers only rates of convergence without boundary conditions or with a finite time horizon, whereas finite-difference approximations for Bellman equations with cylindrical domains were treated in [6]. Using analytic (nonlinear partial differential equation (PDE)) techniques, the rates of convergence issues have been studied in great generality in the aforementioned references.

In reference to these developments, the current paper deals with regime- switching models involving stopping times in the cost criteria. There are added difficulties due to the Markovian coupling and boundary conditions. First, owing to the presence of the switching mechanism, we need to treat a number of cost and value functions. In lieu of treating a single HJB equation as in the case of controlled diffusions, we have to deal with a system of coupled PDEs. Although the convergence of Markov chain approximation to the controlled switching diffusion processes was obtained in [20], there appears to be no rate of convergence result for such controlled switching diffusions to date.

The numerics of stochastic controls of diffusions with a stopping time is difficult to study due to the added boundary conditions. As illustrated in [17, p. 278] (see also Figure 5.1 and section 5.3 of this paper), because of the appearance of the boundary, a "tangency" problem may arise. Thus an added assumption is commonly used for the convergence of the algorithm to avoid the tangency problem. It is difficult to get convergence rates even for the expectation of a stopping time (a stochastic control problem with a constant running cost). To overcome the difficulty due to the stopping time, in this work, we generalize the traditional view of cost and value functions by noting the explicit dependence on another parameter, namely, the boundary. Thus, in addition to the dependence on the state, we regard the value functions as functions of the boundary as well. As an immediate consequence, a nice property, namely, continuous dependence on the boundary of the value function, follows. This continuity enables us to take a closer look at the value function and to get a much better understanding of the numerical approximation scheme using Markov chain approximation.

This paper presents a first attempt using a probabilistic approach to treat such rates of convergence problems. In addition, in contrast to the significant developments in the literature using PDE methods for approximation of controlled diffusions, there do not yet appear to be any PDE results to date for rates of convergence of numerical solutions for controlled switching diffusions, to the best of our knowledge. Although some of our working conditions such as the one-dimensional continuous state variable, nondegenerate diffusions, and control only on the drift may be seemingly strong, they are adequate as the starting point for using this new approach to treat the rates of convergence problems.

The classical Markov chain approximation methods developed in [15, 16, 17] use weak convergence methods for proving the convergence. In the weak convergence setup, various processes (the approximation sequences and the original stochastic processes) may live in different probability spaces. To study the convergence rates, comparisons of different processes are needed. To facilitate the study, we use a strong approximation technique and imbed all the stochastic processes in the same space.

Owing to the Markov chain scheme used, we have to face the problem of "adaptation" of continuous controls to discrete controls. The rate of convergence is ascertained by obtaining upper and lower bounds, respectively. To get lower bounds, adaptation from discrete control to continuous control is needed. Similar techniques were used in [19]. In contrast to the rates of convergence study to date, to obtain upper bounds, we adapt continuous controls to discrete controls by using some properties of relaxed controls.

The rest of the paper is arranged as follows. The precise formulation of the problem is presented next. Also introduced is the notion of weak approximation. Section 3 proceeds with strong approximation using relaxed control representation. Section 4 obtains upper and lower bounds of the approximate value functions. Section 5 contains discussions on several issues. First some specific models are considered. Then another assumption is proposed, which is itself an interesting PDE problem leading to the verification of the condition on the cost function. Somewhat remarkable, as a by-product, we prove that the anticipated tangency problem will not happen in the sense of convergence in probability and in L^1. Finally, an appendix that contains the proofs of a number of auxiliary results is provided. The results in the appendix are of interest in their own right not only for the rate of convergence study but also for other stochastic control problems as well.

2. Problem formulation. Suppose that $\mathcal{M} = \{1, \ldots, m_0\}$ is a finite set and α_t is a continuous-time Markov chain with state space $\mathcal{M}$ and generator $Q = (q_{ij}) \in \mathbb{R}^{m_0 \times m_0}$ satisfying $q_{ij} \geq 0$ for $i \neq j$ and $\sum_{j=1}^{m_0} q_{ij} = 0$ for each $i \in \mathcal{M}$. Consider a pair of random processes (X_t, α_t) in the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W_\cdot, \alpha_\cdot)$, which satisfies

$$\begin{cases} X_t = x + \int_0^t b_{\alpha_s}(X_s, u_s)ds + \int_0^t \sigma_{\alpha_s}(X_s)dW_s, \\ \alpha_t \ \text{ is a continuous-time Markov chain with } \ \alpha_0 = i, \end{cases} \tag{2.1}$$

where W_t is a standard Brownian motion independent of the Markov chain α_t. For a given $B > 0$, define a stopping time as

$$\tau_B^{x,i,u} = \inf\{t : X^{x,i,u}(t) \notin (-B, B)\}.$$

Our objective is to choose the control $u_\cdot$ so as to minimize the expected cost function

$$\begin{cases} J_i^B(x, u) = E\left[\int_0^{\tau_B^{x,i,u}} f_{\alpha_s}(X_s, u_s)ds\right] & \forall x \in (-B, B),\ i \in \mathcal{M}, \\ J_i^B(x, u) = 0 & \forall x \notin (-B, B),\ i \in \mathcal{M}, \end{cases} \tag{2.2}$$

where for each $i \in \mathcal{M}$, $f_i(\cdot, \cdot)$ is an appropriate function representing the running cost function. For each $i \in \mathcal{M}$, the value function is given by

$$V_i^B(x) = \inf_{u \in \mathcal{U}} J_i^B(x, u), \tag{2.3}$$

where $\mathcal{U}$ is the space of all $\mathcal{F}_t$-adapted controls taking values on a compact set U, which is referred to as an ordinary control space. Formally, the value functions satisfy a system of $m_0 = |\mathcal{M}|$ HJB equations,

$$\begin{cases} \inf_{r \in U}\{L^r V_i^B(x) + f_i(x, r)\} = 0 & \forall x \in (-B, B), i \in \mathcal{M}, \\ V_i^B(x) = 0 & \forall x \notin (-B, B), i \in \mathcal{M}, \end{cases} \tag{2.4}$$

where the operator L with parameter $r \in U$ on $\{\varphi_i \in C^2(\mathbb{R}) : i \in \mathcal{M}\}$ is

$$L^r \varphi_i(x) = \frac{1}{2}\sigma_i^2(x)\frac{d^2\varphi_i(x)}{dx^2} + b_i(x,r)\frac{d\varphi_i(x)}{dx} + \sum_{j\in\mathcal{M}} q_{ij}\varphi_j(x).$$

Note that the numerical approximation of the Markov chain approximation methods for regime-switching diffusions has been developed in [20]. However, the rates of convergence of the approximation has not been studied yet to the best of our knowledge.

Remark 2.1. Owing to the presence of the regime switching, instead of one cost function and one value function as in the setup of controlled diffusion processes, a collection of cost functions and value functions must be taken into consideration. The inclusion of the random switching process enables us to incorporate various applications involving random environment and other stochastic behaviors. On the other hand, the coupling due to the switching process causes much difficulty in the analysis as well as numerical approximation. In the above, the explicit dependence of V on B is given, which is an extension of the traditional view of the value function. As a motivation, we consider the first exit time from $(-B, B)$. Suppose the diffusion is given by (2.1) and the objective function is (2.2) with $f \equiv 1$. The problem is to find the optimal control of the mean first exit time. A reference to this problem can be found in [8, p. 158].

For the controlled switching diffusion, in [20], we constructed a locally consistent, discrete-time, controlled Markov chain (see [20] for a definition of local consistency; see also [17] for the diffusion counterpart). We briefly explain the idea and refer the reader to the aforementioned references for further reading. Let $h > 0$ be a discretization parameter. Define $S_h = \{x : x = kh, k = 0, \pm 1, \pm 2, \ldots\}$. Let $\{(\xi_n^h, \alpha_n^h), n < \infty\}$ be a controlled discrete-time Markov chain on a discrete state space $S_h \times \mathcal{M}$ with transition probabilities from a state $(x, \imath) \in \mathcal{M}$ to another state $(y, \jmath) \in \mathcal{M}$, denoted by $p^h((x,\imath),(y,\jmath)|r)$ for $r \in U$. For notational simplicity, we denote

$$(-B,B)_h = (-B,B) \cap S_h, \quad [-B,B]_h = (-B,B)_h \cup \{B,-B\}. \tag{2.5}$$

Note that in the traditional setup for controlled diffusions, the cost function and value function are written as $J(x,u)$ and $V(x)$, respectively. Here, we modify the notion by introducing the dependence of the boundary of the state, namely, B. Such a notation will facilitate the analysis in the use of boundary perturbations.

For convenience in the analysis to follow, we present some definitions first. Let $G = [-B-\varepsilon, B+\varepsilon]$ for some small $\varepsilon > 0$. For a real-valued function $\phi(\cdot)$ on G and a collection of functions $\{\psi_i(\cdot,\cdot), i \in \mathcal{M}\}$ on $G \times U$, and for $\gamma \in (0,1]$, define

$$\begin{aligned}
|\phi|_\infty &= \max_{x\in G} |\phi(x)|,\\
|\psi_i|_\infty &= \max_{(x,r)\in G\times U} |\psi_i(x,r)|,\\
|\psi|_\infty &= \max_{i\in\mathcal{M}} |\psi_i|_\infty,\\
|\phi|_\gamma &= \sup_{x,y\in G, x\neq y} \frac{|\phi(x)-\phi(y)|}{|x-y|^\gamma},\\
|\psi_i|_\gamma &= \sup_{r\in U} |\psi_i(\cdot,r)|_\gamma,\\
|\psi|_\gamma &= \max_{i\in\mathcal{M}} |\psi_i|_\gamma.
\end{aligned} \tag{2.6}$$

Throughout the paper, we use K to denote a generic positive constant and we use K_t to denote a generic positive constant depending on t. Throughout the paper, we use the following assumptions.

(H1) For functions $b(\cdot)$, $\sigma(\cdot)$, $f(\cdot)$, there is a $\rho \in (0,1]$ such that

$$|\sigma|_\infty + |b|_\infty + |\sigma|_1 + |b|_1 + |f|_\infty + |f|_\rho \le K < \infty. \tag{2.7}$$

(H2) $\sigma_i(x) > 0 \; \forall (i,x) \in \mathcal{M} \times G$.

(H3) Q is irreducible in the sense that the system of equations

$$\begin{cases} \nu Q = 0, \\ \displaystyle\sum_{i\in\mathcal{M}} \nu_i = 1 \end{cases} \tag{2.8}$$

has a unique solution $\nu = (\nu_1, \dots, \nu_{m_0}) \in \mathbb{R}^{1\times m_0}$ satisfying $\nu_i > 0$ for each $i \in \mathcal{M}$.

Define a finite difference operator

$$L_h^r \varphi_i(x) = \frac{1}{2}\sigma_i^2(x)\Delta^h\varphi_i(x) + b_i(x,r)\delta^h\varphi_i(x) + \sum_{j\in\mathcal{M}} q_{ij}\varphi_j(x),\ i \in \mathcal{M}, \tag{2.9}$$

where

$$\begin{aligned} \delta^h\varphi_i(x) &= \frac{\varphi_i(x+h) - \varphi_i(x-h)}{2h}, \\ \Delta^h\varphi_i(x) &= \frac{\varphi_i(x+h) + \varphi_i(x-h) - 2\varphi_i(x)}{h^2}. \end{aligned}$$

Then, $\bar{V}_i^{B,h}(x)$, the discretization of $V_i^B(x)$ with step size $h > 0$, is the solution of

$$\begin{cases} \inf_{r\in U}\{L_h^r \bar{V}_i^{B,h}(x) + f_i(x,r)\} = 0 & \forall x \in (-B,B)_h,\ i \in \mathcal{M}, \\ \bar{V}_i^{B,h}(x) = 0 & \forall x \notin (-B,B)_h,\ i \in \mathcal{M}, \end{cases} \tag{2.10}$$

where $(-B,B)_h$ is as given in (2.5). For notational convenience, define

$$\begin{aligned} p_i^{h,\pm}(x,r) &= \frac{1}{2} \pm \frac{b_i(x,r)h}{2\sigma_i^2(x)}, \\ p_{ij}^h(x) &= \frac{q_{ij}h^2}{\sigma_i^2(x)} \quad \forall j \ne i, \\ p_{ii}^h(x) &= 1 + \frac{q_{ii}h^2}{\sigma_i^2(x)}, \\ \Delta t_i^h(x) &= \frac{h^2}{\sigma_i^2(x)}. \end{aligned} \tag{2.11}$$

Using (2.11) and rearranging (2.10) lead to

$$\begin{aligned} \bar{V}_i^{B,h}(x) = \inf_{r\in U} \Bigg\{ & \bar{p}_i^{h,+}(x,r)\bar{V}_i^{B,h}(x+h) + \bar{p}_i^{h,-}(x,r)\bar{V}_i^{B,h}(x-h) \\ & + \sum_{j\ne i} \bar{p}_{ij}^h(x)\bar{V}_j^{B,h}(x) + f_i(x,r)\Delta\bar{t}_i^h(x) \Bigg\}, \end{aligned} \tag{2.12}$$

where

$$
\begin{aligned}
\bar{p}_i^{h,\pm}(x,r) &= \frac{\sigma_i^2(x) \pm hb_i(x,r)}{2(\sigma_i^2(x) - h^2 q_{ii})} = p_{ii}^h(x) p_i^{h,\pm}(x,r) + O(h^3), \\
\bar{p}_{ij}^h(x) &= \frac{q_{ij}h^2}{\sigma_i^2(x) - q_{ii}h^2} = p_{ij}^h(x) + O(h^4) \quad \forall j \neq i, \\
\Delta \bar{t}_i^h(x) &= \frac{h^2}{\sigma_i^2(x) - q_{ii}h^2} = p_{ii}^h(x) \Delta t_i^h(x) + O(h^4).
\end{aligned} \tag{2.13}
$$

Note that (2.12) is a dynamic programming equation for a controlled Markov chain with transition probabilities (2.13); see the details in [17, 20]. It is well known that there exists a unique viscosity solution V_i^B of (2.4) by [22, Theorem A.24], and the solution $\bar{V}_i^{B,h}$ of (2.10) converges to the unique viscosity solution V_i^B as $h \to 0$.

Remark 2.2. In what follows, we call the Markov chain generated by transition probability (2.13) the weak approximation. The reason for the use of "weak approximation" is that, generally, the approximating process and the given original process $X_\cdot^{x,u}$ lie in different probability spaces and the convergence under consideration is in the weak sense; see [17]. We use $\bar{V}_i^{B,h}$ to denote the corresponding value function under weak approximation. Later, $V_i^{B,h}$ will be used to denote the value function under strong approximation. By strong approximation, we mean that all the approximation processes are constructed in the same probability space on which $X_\cdot$ is defined; see [3] and the references therein for strong approximation of stochastic processes.

In this paper, we take up the issue of studying the rates of convergence. This may be viewed as the rate of convergence of the dynamic programming equation (2.12). Nevertheless, there are some specifics when Markov chain approximations are used for the approximation of the associate control problem. For $\gamma \in (2, 3]$, consider

$$
\begin{aligned}
\bar{p}_i^{h,\pm}(x,r) &= p_{ii}^h(x) p_i^{h,\pm}(x,r) + O(h^\gamma) \\
&= \frac{1}{2} \pm \frac{b_i(x,r)h}{2\sigma_i^2(x)} + \frac{q_{ii}h^2}{2\sigma_i^2(x)} + O(h^\gamma), \\
\bar{p}_{ij}^h(x) &= p_{ij}^h(x) + O(h^{\gamma+1}) \\
&= \frac{q_{ij}h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}) \quad \forall j \neq i, \\
\Delta \bar{t}_i^h(x) &= p_{ii}^h(x) \Delta t_i^h(x) + O(h^{\gamma+1}) \\
&= \frac{h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}).
\end{aligned} \tag{2.14}
$$

We assume that the transition probabilities for the Markov chain α_t are given by

$$
P(\alpha_{t+\Delta} = j | \alpha_t = i) = q_{ij}\Delta + O(\Delta^2) \quad \forall j \neq i. \tag{2.15}
$$

As worked out in [20], our approximating Markov chain has two components. One component is an approximation to the diffusion, whereas the other component keeps track of the discrete events. In the usual setup of the Markov chain, in lieu of $O(\Delta^2)$ in the last term on (2.15), $o(\Delta)$ is normally used. If we use $o(\Delta)$, we will still get the same error estimates for the HJB (2.4) and the dynamic programming equation (2.12) for the approximated value functions. Owing to the uniqueness of the solution, we can replace $o(\Delta)$ by $O(\Delta^2)$. This is convenient for our work in what follows. To proceed, our problem is stated as follows.

Problem 2.3. Find the upper bound of

$$|\bar{V}^{B,h} - V^B|_\infty,$$

where V^B is the value function (2.3) with the underlying two-component controlled process (2.1), and $\bar{V}^{B,h}$ is the value function corresponding to the Markov chain approximation with the two-component Markov chain generated by transition probability (2.14) satisfying the dynamic programming equation (2.12).

3. Strong Markov chain approximation under relaxed controls. To facilitate the analysis, we introduce the relaxed control representation; see [17]. We consider the same optimal control problem by extending real-valued control space $\mathcal{U}$ to the measure-valued control space Γ. It allows us to work with convergence analysis in relaxed control space. Moreover, we construct strong approximation under relaxed controls in the same probability space of solutions of the controlled switching diffusions, which have approximately the same value functions as in the weak approximation given in the previous section, and which make comparisons of different functions possible within the same probability space in the subsequent sections.

3.1. Relaxed control formulation.

DEFINITION 3.1. *Let $\mathcal{B}(U\times[0,\infty))$ be the σ-algebra of Borel subsets of $U\times[0,\infty)$. An admissible relaxed control (or deterministic relaxed control) $m(\cdot)$ is a measure on $\mathcal{B}(U\times[0,\infty))$ such that $m(U\times[0,t]) = t$ for each $t \geq 0$. Given a relaxed control $m(\cdot)$, there is an $m_t(\cdot)$ such that $m(drdt) = m_t(dr)dt$. In fact, we can define $m_t(B) = \lim_{\delta\to 0}\frac{m(B\times[t-\delta,t])}{\delta}$ for $B\in\mathcal{B}(U)$.*

With the given probability space, we say that $m(\cdot)$ is an admissible relaxed (stochastic) control for $(W(\cdot),\alpha(\cdot))$, or $(m(\cdot),W(\cdot),\alpha(\cdot))$ is admissible if $m(\cdot,\omega)$ is a deterministic relaxed control with probability 1 *and if $m(A\times[0,t])$ is $\mathcal{F}_t$-adapted for all $A\in\mathcal{B}(U)$. There is a derivative $m_t(\cdot)$ such that $m_t(\cdot)$ is $\mathcal{F}_t$-adapted for all $A\in\mathcal{B}(U)$.*

Let $\mathcal{P}(U)$ be the collection of probability measures on $\mathcal{B}(U)$. Then a relaxed control $\{m_t : t > 0\}$ can be considered as an $\mathcal{F}_t$-adapted control taking values in $\mathcal{P}(U)$. Let Γ be the collection of all admissible relaxed controls. Let $\mathcal{U}$ be the collection of admissible U-valued controls, which are sometimes referred to as ordinary controls in contrast to relaxed controls. Then Γ is a convex hull of $\mathcal{U}$. Define

$$\phi(\cdot,\mu) = \int_U \phi(\cdot,r)\mu(dr) \quad \text{for} \quad \mu\in\mathcal{P}(U). \tag{3.1}$$

Let $m(\cdot)$ be a relaxed control, which is an $\mathcal{F}_t$-adapted control taking values in $\mathcal{P}(U)$. The coupled random process $(X_t^{x,i,m},\alpha_t)$ with control $m(\cdot)$ satisfies

$$\begin{cases} X_t = x + \displaystyle\int_0^t b_{\alpha_s}(X_s,m_s)ds + \int_0^t \sigma_{\alpha_s}(X_s)dW_s, \\ \alpha_t \ \text{ is a continuous-time Markov chain generated by } \ Q \ \text{ with } \ \alpha_0 = i. \end{cases} \tag{3.2}$$

For each $i\in\mathcal{M}$, the associated stopping time is given by

$$\tau_B^{x,i,m} = \inf\{t : X^{x,i,m}(t) \notin (-B,B)\},$$

the objective function is given by

$$\begin{cases} J_i^B(x,m) = E\left[\displaystyle\int_0^{\tau_B^{x,i,m}} f_\alpha(X_s,m_s)ds\right] & \forall x\in(-B,B),\ i\in\mathcal{M}, \\ J_i^B(x,m) = 0 & \forall x\notin(-B,B),\ i\in\mathcal{M}, \end{cases} \tag{3.3}$$

and the value function is

$$V_i^B(x) = \inf_{m\in\Gamma} J_i^B(x,m),$$

where Γ is the relaxed control space.

Formally, the above value functions satisfy a system of HJB equations,

$$\begin{cases} \inf_{\mu\in\mathcal{P}(U)} \{L^\mu V_i^B(x) + f_i(x,\mu)\} = 0 & \forall x \in (-B,B),\ i\in\mathcal{M}, \\ V_i^B(x) = 0 & \forall x \notin (-B,B),\ i\in\mathcal{M}, \end{cases} \tag{3.4}$$

where L^μ with $\mu \in \mathcal{P}(U)$ is a natural generalization of L^r with $r\in U$. Since $\mathcal{P}(U)$ is a convex hull of U, the solution of (3.4) is the same as that of (2.4). Hence, the value function under relaxed control is also equal to the value function under ordinary control.

To study the rate of the convergence of the Markov chain approximation, it is equivalent to consider the convergence rate of solutions of dynamic programming equation under relaxed control:

$$\begin{aligned} \bar V_i^{B,h}(x) = \inf_{\mu\in\mathcal{P}(U)} \Bigg\{ & \bar p_i^{h,+}(x,\mu)\bar V_i^{B,h}(x+h) + \bar p_i^{h,-}(x,\mu)\bar V_i^{B,h}(x-h) \\ & + \sum_{j\neq i} \bar p_{ij}^h(x) \bar V_j^{B,h}(x) + f_i(x,\mu)\Delta \bar t_i^h(x) \Bigg\} \quad \forall x\in(-B,B)_h,\ i\in\mathcal{M}, \end{aligned} \tag{3.5}$$

where for given $\gamma\in(2,3]$ and $(x,i)\in(-B,B)_h\times\mathcal{M}$,

$$\begin{aligned} \bar p_i^{h,\pm}(x,\mu) &= p_{ii}^h(x) p_i^{h,\pm}(x,\mu) + O(h^\gamma) \\ &= \frac{1}{2} \pm \frac{b_i(x,\mu)h}{2\sigma_i^2(x)} + \frac{q_{ii}h^2}{2\sigma_i^2(x)} + O(h^\gamma), \\ \bar p_{ij}^h(x) &= p_{ij}^h(x) + O(h^{\gamma+1}) \\ &= \frac{q_{ij}h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}) \quad \forall j\neq i, \\ \Delta \bar t_i^h(x) &= p_{ii}^h(x)\Delta t_i^h(x) + O(h^{\gamma+1}) \\ &= \frac{h^2}{\sigma_i^2(x)} + O(h^{\gamma+1}). \end{aligned} \tag{3.6}$$

Using a relaxed control setup, Problem 2.3 is equivalent to the following problem.

Problem 3.2. Find the upper bound of

$$|\bar V^{B,h} - V^B|_\infty,$$

where V^B is the value function (3.4) with the underlying two-component controlled process (3.2), and $\bar V^{B,h}$ is the piecewise constant interpolation of the value function corresponding to the Markov chain approximation with the two-component Markov chain generated by transition probability (3.6) satisfying the dynamic programming equation (3.5).

3.2. Strong Markov chain approximation.

Construction 3.3 (strong Markov chain approximation). Define a sequence of discrete random variables $\{(x_n^h, \alpha_n^h, m_n^h)\}$ with $\Delta\tau_n^h = \tau_{n+1}^h - \tau_n^h$ in the same probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W., \alpha.)$. For convenience, let us use $(x^h(t), \alpha^h(t), m_t^h)$ to denote the piecewise constant interpolation of $\{(x_n^h, \alpha_n^h, m_n^h)\}$ with $\{\tau_n^h\}$ used, and let the space of such adapted controls be Γ^h. For a given $\mathcal{F}_{\tau_n^h}$-adapted control m^h, $\{(x_n^h, \alpha_n^h)\}$ is a Markov chain defined by the following:

1. Set $x_0^h = x, \alpha_0^h = i, m_0^h = m_0^h(i, x)$ in $\mathcal{F}_0$.
2. Given $(x_n^h, \alpha_n^h, m_n^h) \in \mathcal{F}_{\tau_n^h}$, let $\alpha_{n+1}^h = \alpha(\tau_n^h)$.
   (a) If $\alpha_{n+1}^h = \alpha_n^h$, then

$$\begin{aligned}
\Delta\tau_n^h &= \inf\{t : |b_{\alpha_n^h}(x_n^h, m_n^h)t + \sigma_{\alpha_n^h}(x_n^h)(W(\tau_n^h + t) - W(\tau_n^h))| \geq h\},\\
x_{n+1}^h &= x_n^h + b_{\alpha_n^h}(x_n^h, m_n^h)\Delta\tau_n^h + \sigma_{\alpha_n^h}(x_n^h)(W(\tau_{n+1}^h) - W(\tau_n^h)),\\
m_{n+1}^h &= m_{n+1}^h(\alpha_{n+1}^h, x_{n+1}^h);
\end{aligned}$$

   (b) else if $\alpha_{n+1}^h \neq \alpha_n^h$, then

$$\tau_n^h = 0, \quad x_{n+1}^h = x_n^h, \quad m_{n+1}^h = m_{n+1}^h(\alpha_{n+1}^h, x_{n+1}^h).$$

The above technique is in the spirit of the Skorohod representation; see [9, Theorem A.1] and also [21, Theorem 4.3]. We also note that similar processes are used for the same purpose in [19] and [23].

For each $i \in \mathcal{M}$, let the value function be

$$V_i^{B,h}(x) = \inf_{m^h \in \Gamma^h} \sum_{k=0}^{N_B - 1} f_{\alpha_n^h}(x_n^h, m_n^h)\Delta\tau_n^h \triangleq \inf_{m^h \in \Gamma^h} J_i^{B,h}(x, m^h), \tag{3.7}$$

where $N_B = \inf\{n : x_n^h \notin B^h\}$. The corresponding dynamic programming equation is

$$\begin{aligned}
V_i^{B,h}(x) = \inf_{\mu \in \mathcal{P}(U)} \Bigg\{ &\widetilde{p}_i^{h,+}(x, \mu)V_i^{B,h}(x + h) + \widetilde{p}_i^{h,-}(x, \mu)V_i^{B,h}(x - h)\\
&+ \sum_{j \neq i} \widetilde{p}_{ij}^h(x)V_j^{B,h}(x) + f_i(x, \mu)\Delta\widetilde{t}_i^h(x) \Bigg\}, \ i \in \mathcal{M},
\end{aligned} \tag{3.8}$$

where, owing to Lemma A.4 in the appendix and (2.15), for each $i \in \mathcal{M}$,

$$\begin{aligned}
\widetilde{p}_i^{h,\pm}(x, \mu) &= p_{ii}^h(x)p_i^{h,\pm}(x, \mu) + O(h^3)\\
&= \frac{1}{2} \pm \frac{b_i(x, \mu)h}{2\sigma_i^2(x)} + \frac{q_{ii}h^2}{2\sigma_i^2(x)} + O(h^3),\\
\widetilde{p}_{ij}^h(x) &= p_{ij}^h(x) + O(h^4)\\
&= \frac{q_{ij}h^2}{\sigma_i^2(x)} + O(h^4) \quad \forall j \neq i,\\
\Delta\widetilde{t}_i^h(x) &= p_{ii}^h(x)\Delta t_i^h(x) + O(h^4)\\
&= \frac{h^2}{\sigma_i^2(x)} + O(h^4).
\end{aligned} \tag{3.9}$$

THEOREM 3.4. *Let $V_i^{B,h}(\cdot)$ and $\bar{V}_i^{B,h}(\cdot)$ be the solutions of (3.5) and (3.8), respectively. Then*

$$|V_i^{B,h}(x) - \bar{V}_i^{B,h}(x)| \le Kh^{\gamma-2} \quad \forall (i,x) \in \mathcal{M} \times (-B,B)_h$$

for some constant K.

Proof. It is enough to prove that

$$|J_i^{B,h}(x,m) - \bar{J}_i^{B,h}(x,m)| \le Kh^{\gamma-2}$$

for any feedback control $m(i,x)$ adopted by weak approximation (3.3) and strong approximation (3.7). Thus we have for each $i \in \mathcal{M}$,

$$\begin{aligned} J_i^{B,h}(x,m) = {} & \widetilde{p}_i^{h,+}(x,m) J_i^{B,h}(x+h,m) + \widetilde{p}_i^{h,-}(x,m) J_i^{B,h}(x-h,m) \\ & + \sum_{j\neq i} \widetilde{p}_{ij}^h(x) J_j^{B,h}(x,m) + f_i(x,m)\Delta \widetilde{t}_i^h(x) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} \bar{J}_i^{B,h}(x,m) = {} & \bar{p}_i^{h,+}(x,m) \bar{J}_i^{B,h}(x+h,m) + \bar{p}_i^{h,-}(x,m) \bar{J}_i^{B,h}(x-h,m) \\ & + \sum_{j\neq i} \bar{p}_{ij}^h(x) \bar{J}_j^{B,h}(x,m) + f_i(x,m)\Delta \bar{t}_i^h(x). \end{aligned} \tag{3.11}$$

Subtracting (3.11) from (3.10) and setting $e_i^h(x) = J_i^{B,h}(x,m) - \bar{J}_i^{B,h}(x,m)$,

$$\begin{aligned} e_i^h(x) = {} & p_{ii}^h(x) p_i^{h,+}(x,m) e_i^h(x+h) + p_{ii}^h(x) p_i^{h,-}(x,m) e_i^h(x-h) \\ & + \sum_{j\neq i} p_{ij}^h(x) e_j^h(x) + k_i^h(x), \end{aligned} \tag{3.12}$$

with $|k_i^h(x)| \le Kh^\gamma$ and $e_i^h(\pm B) = 0$.

By the probability representation (3.12), one can show that e^h satisfies the comparison result with respect to the running cost k^h. That is, let e^h and $\tilde{e}^h$ be the solution of (3.12) with different running cost k^h and $\tilde{k}^h$, respectively; then, for all (i,x),

$$k_i^h(x) \le \tilde{k}_i^h(x) \text{ implies } e_i^h(x) \le \tilde{e}_i^h(x).$$

Hence, to estimate $|e^h|_\infty = \max_{i,x} |e_i^h(x)|$, it suffices to take $k_i^h(x) \equiv Kh^\gamma$ to get an upper bound of $|e^h|_\infty$.

Let $i(x) \in \mathcal{M}$ be such that $e_{i(x)}^h(x) = \max_{i\in\mathcal{M}} e_i^h(x) \triangleq e^h(x)$. Using $i = i(x)$ in (3.12), it follows that

$$\begin{aligned} e^h(x) \le {} & p_{ii}^h(x) p_i^{h,+}(x,m) e^h(x+h) + p_{ii}^h(x) p_i^{h,-}(x,m) e^h(x-h) \\ & + \sum_{j\neq i} p_{ij}^h(x) e^h(x) + Kh^\gamma. \end{aligned}$$

Using $p_{ii}^h(x) = 1 - \sum_{j\neq i} p_{ij}^h(x)$, one can rearrange the above inequality to get

$$0 \le e^h(x) \le p_i^{h,+}(x,m) e^h(x+h) + p_i^{h,-}(x,m) e^h(x-h) + Kh^\gamma.$$

Once again, by the comparison result, one can show $e^h(x) \le \hat{e}^h(x)$, where $e^h(x)$ satisfies

$$\hat{e}^h(x) = p_i^{h,+}(x,m)\hat{e}^h(x+h) + p_i^{h,-}(x,m)\hat{e}^h(x-h) + Kh^\gamma.$$

Note that $p_i^{h,\pm}(x,m)$ satisfies the conditions of Lemma A.1 from (2.11); hence the result of Lemma A.1 in the appendix implies $|\hat{e}^h|_\infty \le Kh^{\gamma-2}$. ◻

Thanks to Theorem 3.4 and the triangle inequality, we have

$$|\bar{V}^{B,h} - V^B|_\infty \le |\bar{V}^{B,h} - V^{B,h}|_\infty + |V^{B,h} - V^B|_\infty \le Kh^{\gamma-2} + |V^{B,h} - V^B|_\infty.$$

Thus, Problem 3.2 can be reduced to following problem.

Problem 3.5. Find the upper bound of

$$|V^{B,h} - V^B|_\infty,$$

where V^B is the value function (3.4) with the underlying two-component controlled process (3.2), and $V^{B,h}$ is the value function corresponding to the strong Markov chain approximation with the two-component Markov chain generated by Construction 3.3 satisfying the dynamic programming equation (3.8).

To proceed, define

$$z^h(t) \triangleq \max\{j : \tau_j^h \le t\}, \quad \tau_t^h \triangleq \tau_{z^h(t)}^h. \tag{3.13}$$

Note

$$\begin{aligned} E\left[\int_0^t \mathbb{1}_{\{\alpha_s^h \ne \alpha_s\}} ds\right] &= E\left[\int_0^t E[\mathbb{1}_{\{\alpha_s^h \ne \alpha_s\}} | \mathcal{F}_{\tau_s^h}] ds\right] \\ &= E\left[\int_0^t E[(-q_{ii})(s-\tau_s^h) + O((s-\tau_s^h)^2) | \mathcal{F}_{\tau_s^h}]\right] ds \quad \text{(by (2.15))} \\ &\le E\left[\int_0^t (-q_{ii}) \left(\frac{h^2}{\min\{\sigma^2\}} + O(h^4)\right) ds\right] \quad \text{(by Lemma A.4)} \\ &\le K_t h^2. \end{aligned}$$

Then an immediate consequence of α^h constructed in the same probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W_\cdot, \alpha_\cdot)$ as in Construction 3.3 is

$$E\left[\int_0^t \mathbb{1}_{\{\alpha_s^h \ne \alpha_s\}} ds\right] \le K_t h^2. \tag{3.14}$$

LEMMA 3.6. *Let the discrete random sequence* $\{(x_n^h, \alpha_n^h, m_n^h)\}$ *be defined in the same probability space* $(\Omega, \mathcal{F}, \mathcal{F}_t, P, W_\cdot, \alpha_\cdot)$ *as in Construction* 3.3. *Suppose that there exists an* $m \in \Gamma$ *such that*

$$x^h(t) = x + \int_0^{\tau_t^h} b_{\alpha_s^h}(x^h(s), m_s) ds + \int_0^{\tau_t^h} \sigma_{\alpha_s^h}(x^h(s)) dW_s$$

and

$$X_t = x + \int_0^t b_{\alpha_s}(X_s, m_s) ds + \int_0^t \sigma_{\alpha_s}(X_s) dW_s.$$

Then, for all $\theta \in (0, 1]$,

$$E|x^h(t) - X_t|^\theta \le K_t h^\theta.$$

Proof. Taking the difference of the above two processes, we arrive at

$$\begin{aligned}
E|x^h(t) - X_t|^2 \le 5E\left|\int_0^{\tau_t^h} b_{\alpha_s^h}(x^h(s), m_s) - b_{\alpha_s}(x^h(s), m_s)ds\right|^2 \\
&+ 5E\left|\int_0^{\tau_t^h} \sigma_{\alpha_s^h}(x^h(s))dW_s - \int_0^{\tau_t^h} \sigma_{\alpha_s}(x^h(s))dW_s\right|^2 \\
&+ 5E\left|\int_0^{\tau_t^h} b_{\alpha_s}(x^h(s), m_s) - b_{\alpha_s}(X_s, m_s)ds\right|^2 \\
&+ 5E\left|\int_0^{\tau_t^h} \sigma_{\alpha_s}(x^h(s))dW_s - \int_0^{\tau_t^h} \sigma_{\alpha_s}(X_s)dW_s\right|^2 \\
&+ 5E\left|\int_{\tau_t^h}^t b_{\alpha_s}(X_s, m_s)ds + \int_{\tau_t^h}^t \sigma_{\alpha_s}(X_s)dW_s\right|^2 \\
&\triangleq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

Then

$$\begin{aligned}
I_1 &\le E\left|\sum_{j=0}^{z^h(t)-1} \int_{\tau_j^h}^{\tau_{j+1}^h} |b_{\alpha_s^h}(x^h(s), m_s) - b_{\alpha_s}(x^h(s), m_s)|ds\right|^2 \\
&\le KE\left|\sum_{j=0}^{z^h(t)-1} \int_{\tau_j^h}^{\tau_{j+1}^h} \mathbb{1}_{\{\alpha_s^h \neq \alpha_s\}} ds\right|^2 \\
&= KE\left|\int_0^{\tau_t^h} \mathbb{1}_{\{\alpha_s^h \neq \alpha_s\}} ds\right|^2 \\
&\le KE\int_0^t \mathbb{1}_{\{\alpha_s^h \neq \alpha_s\}} ds \cdot t \\
&\le K_t h^2 \quad \text{(by (3.14))}.
\end{aligned}$$

Likewise,

$$I_2 \le K_t h^2.$$

Next, by using the Lipschitz condition of b and σ,

$$I_3 \le K_t E\int_0^t |x^h(s) - X_s|^2 ds$$

and

$$I_4 \le KE\int_0^t |x^h(s) - X_s|^2 ds.$$

The last estimate is followed by $t-\tau_t^h \le \Delta\tau^h_{z^h(t)}$, and hence

$$I_5 \le K\Big(\frac{h^2}{\min\{\sigma^2\}} + O(h^4)\Big) \le Kh^2.$$

Putting all the above estimates together, we obtain

$$E|x^h(t) - X_t|^2 \le K_t h^2 + K(t+1)E\int_0^t |x_s^h - X_s|^2 ds.$$

Applying Gronwall's inequality yields

$$E|x^h(t) - X_t|^2 \le K_t e^{Kt^2+Kt}\cdot h^2 \triangleq K_t h^2.$$

Then Jensen's inequality leads to the completion of the proof. □

By applying Lemma 3.6 with $m_t \equiv m_t^h$, it immediately follows that

$$E|x^h(t) - X^{i,x,m^h}(t)|^\theta \le K_t h^\theta \quad \forall \theta \in (0,1].$$

4. Convergence rate. The following assumption will be mentioned clearly if it is needed in subsequent assertions.

(H4) For any $\mathcal{F}_t$-adapted process $m(\cdot)$, there exists constant K such that

$$|J_i^B(x,m) - J_i^B(y,m)| \le K|x-y|^\rho.$$

THEOREM 4.1. *Assume* (H4). *For any $m \in \Gamma$, there exists $m^h \in \Gamma^h$ such that*

$$|J_i^{B,h}(x,m^h) - J_i^B(x,m)| \le Kh^{\frac{1}{2}\wedge\rho}. \tag{4.1}$$

Proof. Without loss of generality, for each $i \in \mathcal{M}$, we assume f_i is a nonnegative function. Otherwise, set $f_i = f_i^+ - f_i^-$. Then

$$\begin{aligned}
&|J_i^{B,h}(x,m^h) - J_i^B(x,m)|\\
&= \left|E\int_0^{\tau_B^h} f_{\alpha_s^h}(x_s^h, m_s^h)ds - E\int_0^{\tau_B} f_{\alpha_s}(X_s, m_s)ds\right|\\
&\le \left|E\int_0^{\tau_B^h} f^+_{\alpha_s^h}(x_s^h, m_s^h)ds - E\int_0^{\tau_B} f^+_{\alpha_s}(X_s, m_s)ds\right|\\
&\qquad + \left|E\int_0^{\tau_B^h} f^-_{\alpha_s^h}(x_s^h, m_s^h)ds - E\int_0^{\tau_B} f^-_{\alpha_s}(X_s, m_s)ds\right|.
\end{aligned}$$

Consequently, we can prove Theorem 4.1 for nonnegative functions f_i^+ and f_i^- separately.

In what follows, we divide the work into three steps. Step 1 constructs a relaxed control $m^h \in \Gamma^h$ with certain properties. Step 2 derives a lower bound on $J_i^{B,h}(x,m^h) - J_i^B(x,m)$, and step 3 further obtains an upper bound.

Step 1. Given $(\alpha_n^h, x_n^h) \in \mathcal{F}_{\tau_n^h}$ and $\alpha_n^h \ne \alpha_{n-1}^h$, we construct $\{m_n^h\} \in \Gamma^h$ from $m \in \Gamma$ as follows:

$$m_n^h(A) = \frac{E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_{n+1}^h} m_t(A)dt\right]}{E_{\tau_n^h}[\Delta\tau_n^h]} \quad \forall A \in \mathcal{B}(U). \tag{4.2}$$

Note that $m_n^h(\cdot)$ can be considered as an averaged occupation measure of $m(\cdot)$ within the interval $[\tau_n^h, \tau_{n+1}^h)$. We establish the existence of $m_n^h \in \mathcal{P}(U)$ as follows. Given $\mathcal{F}_{\tau_n^h}$ and $\mu \in \mathcal{P}(U)$, define

$$\Delta\tau_n^h(\mu) = \inf\{t : |b_{\alpha_n^h}(x_n^h, \mu)t + \sigma_{\alpha_n^h}(x_n^h)(W(\tau_n^h + t) - W(\tau_n^h))| \geq h\},$$

$$p(\mu, A) = \frac{E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_n^h+\Delta\tau_n^h(\mu)} m_t(A)dt\right]}{E_{\tau_n^h}[\Delta\tau_n^h(\mu)]} \quad \forall A \in \mathcal{B}(U).$$

Then $\mu \mapsto p(\mu, \cdot)$ is a mapping from a compact convex set $\mathcal{P}(U)$ to a subset of itself with closed and convex nonempty images. Kakutani's fixed point theorem implies that there exists $\mu^* \in \mathcal{P}(U)$ such that $p(\mu^*, \cdot) = \mu^*(\cdot)$, and μ^* can be defined to be $m_n^h(\cdot)$.

Therefore, we can use $J_i^{B,h}(x, m)$ to denote $J_i^{B,h}(x, m^h(\cdot))$, where $m^h(\cdot)$ is a piecewise constant interpolation of $\{m_n^h\}$ given by (4.2). Also, it can be verified for a function $\varphi = b_i, f_i$ that

$$\begin{aligned} E_{\tau_n^h}[\varphi(x_n^h, m_n^h)\Delta\tau_n^h] &= E_{\tau_n^h}\left[\int_U \varphi(x_n^h, r)m_n^h(dr) \cdot \Delta\tau_n^h\right] \\ &= E_{\tau_n^h}\left[\int_U \varphi(x_n^h, r)\frac{E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_{n+1}^h} m_t(dr)dt\right]}{E_{\tau_n^h}[\Delta\tau_n^h]}\Delta\tau_n^h\right] \\ &= E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_{n+1}^h}\int_U \varphi(x_n^h, r)m_t(dr)dt\right] \\ &= E_{\tau_n^h}\left[\int_{\tau_n^h}^{\tau_{n+1}^h} \varphi(x_n^h, m_t)dt\right]. \end{aligned} \tag{4.3}$$

In (4.3), the second line follows from the definition given in (4.2); the third line is a consequence of Fubini's theorem and the definition in (3.1) together with the measure $p(\mu, A)$ defined in the previous paragraph; the last line follows from the relaxed control definition. Using the constructed $\{m_n^h\}$ with its property (4.3), we can write the strong approximation $x^h(\cdot)$ as

$$\begin{aligned} x^h(t) &= x + \sum_{n=0}^{z^h(t)-1}[b_{\alpha_n^h}(x_n^h, m_n^h)\Delta\tau_n^h + \sigma_{\alpha_n^h}(x_n^h)\Delta W(\tau_n^h)] \\ &= x + \sum_{n=0}^{z^h(t)-1}\left[\int_{\tau_n^h}^{\tau_{n+1}^h} b_{\alpha_s^h}(x_s^h, m_s)ds + \int_{\tau_n^h}^{\tau_{n+1}^h} \sigma_{\alpha_s^h}(x_s^h)dW_s\right] \\ &= x + \int_0^{\tau_t^h} b_{\alpha_s^h}(x_s^h, m_s)ds + \sigma_{\alpha_s^h}(x_s^h)dW_s \end{aligned} \tag{4.4}$$

and

$$X_t = x + \int_0^t b_{\alpha_s}(X_s, m_s)ds + \int_0^t \sigma_{\alpha_s}(X_s)dW_s.$$

By Lemma 3.6, we have

$$E|x^h(t) - X_t|^\theta \le K_t h^\theta \quad \forall \theta \in (0,1]. \tag{4.5}$$

Step 2. In this part, we will obtain a lower bound of $J_i^{B,h}(x,m^h) - J_i^B(x,m)$. Let $B_h = B - h^\theta$ for some $\theta \in (0,1]$, and

$$e_i(x,m) = J_i^{B,h}(x,m^h) - J_i^{B_h}(x,m).$$

Define $\tau = \tau_{B_h}^{i,x,m}$ and $\tau^h = \inf\{t : x^h(t) \notin (-B,B)\}$. Then

$$\begin{aligned} P(\tau^h \le T, \tau^h < \tau) &= P(\tau^h \le T) P(\tau^h < \tau | \tau^h \le T) \\ &\le P(\tau^h \le T) P(|X_{\tau^h} - x^h(\tau^h)| > |B - B_h| \; |\tau^h \le T) \\ &\le P(\tau^h \le T) h^{-\theta} E[|X_{\tau^h} - x^h(\tau^h)| \; |\tau^h \le T] \\ &\le K_T h^{1-\theta} \quad \text{(by (4.5))}. \end{aligned}$$

Rewrite $e_i(x,m)$ as

$$\begin{aligned} e_i(x,m) &= E\left[\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right] \qquad \text{(by (4.3))} \\ &= E\left[\left(\int_0^{\tau^h \wedge \tau} f_{\alpha_s^h}(x_s^h, m_s) - f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h \le T\}}\right] \\ &\quad + E\left[\left(\int_\tau^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds\right) \mathbb{1}_{\{\tau < \tau^h \le T\}}\right] \\ &\quad + E\left[\left(-\int_{\tau^h}^{\tau} f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h \le T\}} \mathbb{1}_{\{\tau^h \le \tau\}}\right] \\ &\quad + E\left[\left(\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h > T, \tau < T\}}\right] \\ &\quad + E\left[\left(\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] \\ &\triangleq I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.6}$$

We can estimate I_i for $i \le 5$ follows:

$$\begin{aligned} I_1 &= E\left[\left(\int_0^{\tau^h \wedge \tau} f_{\alpha_s^h}(x_s^h, m_s) - f_{\alpha_s^h}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h \le T\}}\right] \\ &\quad + E\left[\left(\int_0^{\tau^h \wedge \tau} f_{\alpha_s^h}(X_s, m_s) - f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h \le T\}}\right] \\ &\ge -KE\left[\left(\int_0^{\tau^h \wedge \tau} |x_s^h - X_s|^\rho ds\right) \mathbb{1}_{\{\tau^h \le T\}}\right] - KE\left[\int_0^T \mathbb{1}_{\{\alpha_s^h \ne \alpha_s\}} ds\right] \\ &\ge -K_T h^\rho - K_T h^2 \quad \text{(by (3.14))} \\ &\triangleq -K_T h^\rho, \end{aligned}$$

$$I_2 \geq 0,$$

$$I_3 \geq -K_T P(\tau^h \leq T, \tau^h < T) \geq -K_T h^{1-\theta},$$

and similar to estimation of I_1,

$$I_4 \geq E\left[\left(\int_0^T f_{\alpha_s^h}(x_s^h, m_s)ds - f_{\alpha_s}(X_s, m_s)ds\right) \mathbb{1}_{\{\tau^h > T, \tau < T\}}\right] \geq -K_T h^\rho.$$

Finally, the last term is

$$\begin{aligned}
I_5 &\geq E\left[\left(\int_{\tau_T^h}^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s)ds - \int_{\tau_T^h}^{\tau} f_{\alpha_s}(X_s, m_s)ds\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] \\
&\quad + E\left[\left(\int_0^{\tau_T^h} f_{\alpha_s^h}(x_s^h, m_s)ds - \int_0^{\tau_T^h} f_{\alpha_s}(X_s, m_s)ds\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] \\
&\geq E\left[\left(J^{B,h}_{\alpha(\tau_T^h)}(x^h(\tau_T^h), m(\cdot + \tau_T^h)) - J^{B_h}_{\alpha(\tau_T^h)}(X_{\tau_T^h}, m(\cdot + \tau_T^h))\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] - K_T h^\rho \\
&= E\left[\left(J^{B,h}_{\alpha(\tau_T^h)}(x^h(\tau_T^h), m(\cdot + \tau_T^h)) - J^{B_h}_{\alpha(\tau_T^h)}(x^h(\tau_T^h), m(\cdot + \tau_T^h))\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] \\
&\quad + E\left[\left(J^{B_h}_{\alpha(\tau_T^h)}(x^h(\tau_T^h), m(\cdot + \tau_T^h)) - J^{B_h}_{\alpha(\tau_T^h)}(X_{\tau_T^h}, m(\cdot + \tau_T^h))\right) \mathbb{1}_{\{\tau^h > T, \tau > T\}}\right] - K_T h^\rho \\
&\geq \inf\{e_i(x,m)\} P(\tau^h > T, \tau > T) - K|x^h(\tau_T^h) - X_{\tau_T^h}|^\rho - K_T h^\rho \text{ (by (H4))} \\
&\geq \inf\{e_i(x,m)\} P(\tau^h > T, \tau > T) - K_T h^\rho \text{ (by (4.5))}.
\end{aligned}$$

Thus, putting all the above inequalities together yields

$$e(x,m) \geq \inf\{e_i(x,m)\} P(\tau^h > T, \tau > T) - K_T h^{\rho \wedge (1-\theta)}. \tag{4.7}$$

Inequality (4.7) continues to hold when we take inf on the left-hand side. Therefore,

$$\inf\{e_i(x,m)\} \geq \frac{-K_T h^{\rho \wedge (1-\theta)}}{1 - P(\tau^h > T, \tau > T)}.$$

Note that the left-hand side is T-independent, and T is fixed at the beginning of the proof, which implies

$$\inf\{e_i(x,m)\} \geq -K h^{\rho \wedge (1-\theta)}.$$

This leads to

$$\begin{aligned}
&-K h^{\rho \wedge (1-\theta)} \\
&\leq \inf\{e_i(x,m)\} \\
&\leq J_i^{B,h}(x, m^h) - J_i^{B_h}(x,m) \\
&\leq J_i^{B,h}(x, m^h) - J_i^{B}(x,m) + J_i^{B}(x,m) - J_i^{B_h}(x,m) \\
&\leq J_i^{B,h}(x, m^h) - J_i^{B}(x,m) + E[J_{\alpha(\tau_{B_h})}(X_{\tau_{B_h}}, m)] \text{ (where } X_{\tau_{B_h}} = \pm B_h) \\
&\leq J_i^{B,h}(x, m^h) - J_i^{B}(x,m) + K h^\theta \quad \text{(by Corollary A.3 in the appendix).}
\end{aligned}$$

By rearranging terms, one obtains

$$J_i^{B,h}(x,m^h) - J_i^B(x,m) \geq -Kh^{\rho\wedge(1-\theta)\wedge\theta}.$$

Taking $\theta = \frac{1}{2}$ gives

$$J_i^{B,h}(x,m^h) - J_i^B(x,m) \geq -Kh^{\rho\wedge\frac{1}{2}}.$$

Step 3. In this part, we will obtain an upper bound of $J_i^{B,h}(x,m^h) - J_i^B(x,m)$. The entire proof is parallel to step 2. Let $B^h = B + h^\theta$ for some $\theta \in (0,1]$, and

$$e_i(x,m) = J_i^{B,h}(x,m^h) - J_i^{B^h}(x,m).$$

Define $\tau = \tau_{B^h}^{i,x,m}$ and $\tau^h = \inf\{t : x^h(t) \notin (-B,B)\}$. Then

$$\begin{aligned} P(\tau \leq T, \tau < \tau^h) &= P(\tau \leq T) P(\tau < \tau^h | \tau \leq T) \\ &\leq P(\tau \leq T) P(|X_\tau - x^h(\tau)| > |B - B^h| \; | \tau \leq T) \\ &\leq P(\tau \leq T) h^{-\theta} E[|X_\tau - x^h(\tau)| \; |\tau \leq T] \\ &\leq K_T h^{1-\theta} \quad \text{(by (4.5))}. \end{aligned}$$

Rewrite $e_i(x,m)$ as

$$\begin{aligned} e_i(x,m) &= E\left[\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right] \quad \text{(by (4.3))} \\ &= E\left[\left(\int_0^{\tau^h\wedge\tau} f_{\alpha_s^h}(x_s^h, m_s) - f_{\alpha_s}(X_s,m_s) ds\right) \mathbb{1}_{\{\tau\leq T\}}\right] \\ &\quad + E\left[\left(\int_\tau^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds\right) \mathbb{1}_{\{\tau<T,\tau\leq\tau^h\}}\right] \\ &\quad + E\left[\left(-\int_{\tau^h}^{\tau} f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau^h<\tau\leq T\}}\right] \\ &\quad + E\left[\left(\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau>T,\tau^h<T\}}\right] \\ &\quad + E\left[\left(\int_0^{\tau^h} f_{\alpha_s^h}(x_s^h, m_s) ds - \int_0^\tau f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau>T,\tau^h>T\}}\right] \\ &\triangleq I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.8}$$

Carrying out the estimation as in step 2,

$$I_1 \leq K_T h^\rho, \quad I_2 \leq K_T h^{1-\theta}, \quad I_3 \leq 0,$$

$$I_4 \leq E\left[\left(\int_0^T f_{\alpha_s^h}(x_s^h, m_s) - f_{\alpha_s}(X_s, m_s) ds\right) \mathbb{1}_{\{\tau>T,\tau^h<T\}}\right] \leq K_T h^\rho,$$

and

$$I_5 \leq \sup\{e_i(x,m)\} P(\tau > T, \tau^h > T) + K_T h^\rho.$$

Therefore,

$$\sup\{e_i(x,m)\} \le \frac{K_T h^{\rho\wedge(1-\theta)}}{1-P(\tau^h>T,\tau>T)} = Kh^{\rho\wedge(1-\theta)}.$$

This leads to

$$\begin{aligned} Kh^{\rho\wedge(1-\theta)} &\ge \sup\{e_i(x,m)\} \\ &\ge J_i^{B,h}(x,m^h) - J_i^{B^h}(x,m) \\ &= J_i^{B,h}(x,m^h) - J_i^{B}(x,m) + J_i^{B}(x,m) - J_i^{B^h}(x,m) \\ &\ge J_i^{B,h}(x,m^h) - J_i^{B}(x,m) + E[J_{\alpha(\tau_{B^h})}(\pm B^h, m)] \\ &\ge J_i^{B,h}(x,m^h) - J_i^{B}(x,m) - Kh^{\theta} \text{ (by Corollary A.3)}. \end{aligned}$$

Rearranging terms and taking $\theta = \frac{1}{2}$ lead to

$$J_i^{B,h}(x,m^h) - J_i^{B}(x,m) \le Kh^{\rho\wedge\frac{1}{2}}. \qquad \square$$

The rate of convergence result is a consequence of the above theorem. The result is presented next.

THEOREM 4.2. *Assume* (H4). *The convergence rate is* $(\gamma-2)\wedge\rho\wedge\frac{1}{2}$. *That is,*

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \le Kh^{\frac{1}{2}\wedge\rho\wedge(\gamma-2)} \quad \forall (i,x)\in\mathcal{M}\times G.$$

Proof. Let $m^{h,*}\in\Gamma^h$ be the optimal control such that

$$V_i^{B,h}(x) = J_i^{B,h}(x,m^{h,*}).$$

Existence of the optimal control follows from the property of relaxed controls. Note that $m^{h,*}$ also belongs to Γ, and that by Theorem 4.1,

$$J_i^{B,h}(x,m^{h,*}) - J_i^{B}(x,m^{h,*}) \ge -Kh^{\frac{1}{2}\wedge\rho}.$$

So

$$V_i^{B,h}(x) - V_i^B(x) \ge J_i^{B,h}(x,m^{h,*}) - J_i^{B}(x,m^{h,*}) \ge -Kh^{\frac{1}{2}\wedge\rho}.$$

Let $m^*\in\Gamma$ be optimal control such that

$$V_i^B(x) = J_i^B(x,m^*).$$

Then, by Theorem 4.1 there exists $m^h\in\Gamma^h$ such that

$$J_i^{B,h}(x,m^h) - J_i^{B}(x,m^*) \le Kh^{\frac{1}{2}\wedge\rho},$$

$$V_i^{B,h}(x) - V_i^B(x) \le J_i^{B,h}(x,m^h) - J_i^{B}(x,m^*) \le Kh^{\frac{1}{2}\wedge\rho}.$$

Thus,

$$|V_i^{B,h}(x) - V_i^B(x)| \le Kh^{\frac{1}{2}\wedge\rho}.$$

Combined with Lemma 3.4, it leads to the desired convergence rate. $\square$

Remark 4.3. The rate $(\gamma-2)\wedge\rho\wedge\frac{1}{2}$ is specific for using the Markov chain approximation approach. This is slightly different from the finite difference approach. It is known that the approach of the Markov chain approximation method is useful in the actual computation since little prior information of the HJB equations is needed. In fact, in the actual computation, in lieu of discretizing the PDEs directly, policy improvement methods are used; see [20] for the numerical examples.

5. Discussion. This section is divided into three parts. In the first part, we examine a couple of specific cases for rates of convergence. In the second part, we propose another assumption, (H5), which is a PDE problem leading to the verification of (H4). In the last part, we consider the tangency problem in numerical approximation of stopping time problems.

5.1. Remarks on convergence rates for special cases. In this section, we consider a couple of special cases and discuss related convergence rates. In the first case, the drift b is independent of control. Such systems may arise in certain financial engineering problems. For example, in [24], stock liquidation problems for regime-switching diffusion models are considered, where neither the drift nor the diffusion coefficients depend on control. The objective is to choose the stopping time so as to make profit or cut loss. When the underlying Markov chain has more than two states, no closed-form solution has been found. Thus numerical approximation is a natural choice.

THEOREM 5.1. *Suppose $b_i(x,r) = b_i(x)$. Then, the convergence rate is $(\gamma-2)\wedge\rho\wedge\frac{1}{2}$. That is,*

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \le Kh^{\frac{1}{2}\wedge\rho\wedge(\gamma-2)} \quad \forall (i,x) \in \mathcal{M}\times G.$$

Proof. Owing to Theorem 4.2, it suffices to verify (H4). Observe that the optimal control in this case is simply $u(x,i) = \arg\min_{r\in U} f_i(x,r)$. ($\arg\min_{r\in U} f_i(x,r)$ might be a set. In this case, we may select one member from the set as its representative.) Let $\widetilde{f}_i(x) = \inf_{r\in U} f_i(x,r)$. Then $|\widetilde{f}|_\rho + |\widetilde{f}|_0 \le K$. Hence, it is enough to consider a singleton control space (without control) with cost function

$$J_i^B(x) = E\left[\int_0^{\tau_B^{x,i}} \widetilde{f}_{\alpha_s}(X_s^{x,i})ds\right]$$

and $V_i^B(x) \equiv J_i^B(x)$. Then, Theorem A.2 implies (H4). ∎

In the second case, b and σ are independent of x. We present two motivations. In the first one, consider a regime-switching diffusion model that is linear in the continuous state variable and in which the diffusion coefficient is independent of control. Taking a logarithm transformation, we obtain an equivalent model in which the drift and diffusion coefficients are free of x dependence. The second motivation stems from a controlled Markov chain model that is perturbed by an additional white noise. In both cases, the following result holds.

THEOREM 5.2. *Suppose $b_i(x,r) = b_i(r)$, $\sigma_i(x) = \sigma_i$. Then, the convergence rate is $(\gamma-2)\wedge\rho\wedge\frac{1}{2}$. That is,*

$$|\bar{V}_i^{B,h}(x) - V_i^B(x)| \le Kh^{\frac{1}{2}\wedge\rho\wedge(\gamma-2)} \quad \forall (i,x) \in \mathcal{M}\times G.$$

Proof. Owing to Theorem 4.2, it is enough to verify (H4). For any $\mathcal{F}_t$-adapted process m_t, observe that

$$X_t - Y_t = x - y \quad \text{with probability 1,} \tag{5.1}$$

where $X_t = X_t^{x,i,m}$ and $Y_t = X_t^{y,i,m}$ are the strong solutions of (3.2), that is,

$$X_t = x + \int_0^t b_{\alpha_s}(m_s)ds + \int_0^t \sigma_{\alpha_s} dW_s$$

and

$$Y_t = y + \int_0^t b_{\alpha_s}(m_s)ds + \int_0^t \sigma_{\alpha_s} dW_s.$$

Without loss of generality, assume $B > x > y > -B$. By τ^x and τ^y, we denote the first time X_t and Y_t hit the boundary, respectively. Then

$$\begin{aligned}
&|J_i^B(x,m) - J_i^B(y,m)| \\
&\leq \left| E \int_0^{\tau^x \wedge \tau^y} (f_{\alpha_t}(X_t, m_t) - f_{\alpha_t}(Y_t, m_t))dt \right| \\
&\quad + E[\mathbb{1}_{\{\tau^x < \tau^y\}} |J^B_{\alpha(\tau^x)}(Y(\tau^x), m(\cdot + \tau^x))|] \\
&\quad + E[\mathbb{1}_{\{\tau^x > \tau^y\}} |J^B_{\alpha(\tau^y)}(Y(\tau^y), m(\cdot + \tau^y))|].
\end{aligned}$$

Since $E[\tau^x \wedge \tau^y] < \infty$, the first term is uniformly bounded by $K|x-y|^\rho$ by (5.1). Also, note that

$$P(B - Y(\tau^x) = x - y | \tau^x < \tau^y) = P(B + X(\tau^y) = x - y | \tau^x < \tau^y) = 1.$$

Thus the second and third terms are also bounded by $K|x-y|$ by virtue of Theorem A.2. ∎

5.2. Remark on condition (H4). For simplicity, the discussion is confined to the case of controlled diffusions. The extension to regime-switching diffusion is straightforward. In this paper, we assumed (H4), which could be verified if the following condition (H5) holds. Let $b(\cdot, r)$ and $\sigma(\cdot)$ be Lipschitz continuous and $\sigma(\cdot) > c > 0$ on $G = (-1,1)$, and let U be a compact set.

(H5) Consider

$$\begin{aligned}
&\frac{1}{2}\frac{\partial^2 \Phi}{\partial x^2}\sigma^2(x) + \frac{1}{2}\frac{\partial^2 \Phi}{\partial y^2}\sigma^2(y) + \frac{\partial^2 \Phi}{\partial x \partial y}\sigma(x)\sigma(y) \\
&\quad + \inf_{r\in U}\left\{ \frac{\partial \Phi}{\partial x} b(x,r) + \frac{\partial \Phi}{\partial y} b(y,r) + f(x,y,r) \right\} = 0
\end{aligned} \tag{5.2}$$

for all $(x,y) \in G^2 = G \times G$ with boundary condition

$$\Phi(x,y) = Z_1(x)\mathbb{1}_{\{x\in G\}} + Z_2(y)\mathbb{1}_{\{y \in G\}} \ \forall (x,y) \in \partial G^2, \tag{5.3}$$

where ∂G^2 denotes the boundary of G^2 and Z_1 and Z_2 are smooth functions on G with $Z_1(\pm 1) = Z_2(\pm 1) = 0$. Then, there exists a unique viscosity solution of $C^{0,\rho}(G^2)$.

The following theorem shows that under (H5), condition (H4) holds.

THEOREM 5.3. *Assume* (H5). *Let* $G = (-1,1)$, *and let* $m_t \in \mathcal{F}_t$ *take values in* U. *Consider the stochastic process*

$$X_t^x = x + \int_0^t b(X_t^x, m_t)dt + \sigma(X_t^x)dW_t \tag{5.4}$$

and the related objective function, for $\tau^x = \inf\{t : X_t^x \notin G\}$,

$$J(x,m) = E\left[\int_0^{\tau^x} f(X_t^x, m_t)dt\right].$$

Then

$$|J(x,m) - J(y,m)| \le K|x-y|^\rho.$$

Proof. Define

$$\widetilde{\varphi}(x,m) = \varphi(x,m)\mathbb{1}_{\{x\in G\}}$$

for $\widetilde{\varphi}$ being any of the functions $\widetilde{b}$, $\widetilde{\sigma}$, $\widetilde{f}$. That is, the tilde notation represents a function confined to the set $x \in G$. Then

$$J(x,m) = E\left[\int_0^\infty \widetilde{f}(\widetilde{X}_t^x, m_t)dt\right],$$

where

$$\widetilde{X}_t^x = x + \int_0^t \widetilde{b}(\widetilde{X}_t^x, m_t)dt + \widetilde{\sigma}(\widetilde{X}_t^x)dW_t.$$

Note that $\widetilde{X}_t^x = X_t^x \mathbb{1}_{\{X_t^x \in G\}} + X_{\tau^x}^x \mathbb{1}_{\{X_t^x \notin G\}}$ is a truncated process of X_t^x. Thus, one can write

$$J(x,m) - J(y,m) = E\left[\int_0^\infty (\widetilde{f}(\widetilde{X}_t^x, m_t) - \widetilde{f}(\widetilde{X}_t^y, m_t))dt\right] \triangleq J(x,y,m).$$

Let

$$\Phi(x,y) = \inf_m J(x,y,m).$$

Then, $\Phi(x,y)$ satisfies

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2\Phi}{\partial x^2}\widetilde{\sigma}^2(x) + \frac{1}{2}\frac{\partial^2\Phi}{\partial y^2}\widetilde{\sigma}^2(y) + \frac{\partial^2\Phi}{\partial x\partial y}\widetilde{\sigma}(x)\widetilde{\sigma}(y)\\ &+ \inf_{r\in U}\left\{\frac{\partial\Phi}{\partial x}\widetilde{b}(x,r) + \frac{\partial\Phi}{\partial y}\widetilde{b}(y,r) + \widetilde{f}(x,r) - \widetilde{f}(y,r)\right\} = 0.\end{aligned}$$

The PDE above is equivalent to (5.2) and (5.3), with

$$f(x,y,r) = f(x,r) - f(y,r), \quad Z_1(x) = \inf_m J(x,m), \quad Z_2(y) = \sup_m J(y,m).$$

By (H5), $\Phi(x,y)$ is $C^{0,\rho}(G^2)$. Note that $\Phi(x,x) \equiv 0$ for all $x \in G$. So $|\Phi(x,y) - \Phi(x,x)| \le K|x-y|^\rho$ yields

$$|\Phi(x,y)| \le K|x-y|^\rho.$$

So, $J(x,m) - J(y,m) \ge \Phi(x,y) \ge -K|x-y|^\rho$. By the symmetry of the above inequality, the desired result follows. □

5.3. Tangency problem. For controlled diffusions without switching with a stopping time in the cost function, a natural approach for proving the convergence of a numerical scheme is based on a probabilistic method on a sequence of properly designed Markov chains. One of the difficulties is the so-called tangency problem, as explained in [17, p. 278]. As a result, an assumption is added to complete the proof of convergence in [17]. Using Markov chain approximation techniques, one constructs finite difference schemes for stochastic control problems in which the cost functions involve a first exit time from a bounded region. The true exit time of the diffusion process is replaced by an approximating sequence as well. The sequence of functions converges to a limit function, but the limit may not be the first exit time of the desired diffusion process and it could be tangent to the boundary of the first contact. This problem is referred to as a tangency problem.

Consider X_t of (2.1). Let $x^h(t)$ be a piecewise constant process in Construction 3.3 which approximates X_t in the same probability space. Let τ and τ^h be the first hitting time of X_t and $x^h(t)$ to the boundary. In Figure 5.1, the sequence of τ^h does not converge to τ, even if $x^h_\cdot$ converge to $X_\cdot$. Therefore, it is difficult to estimate the convergence rate $|E\tau^h - E\tau|$, which is the special case of value function (2.2) when the running cost $f(\cdot)$ is a constant.

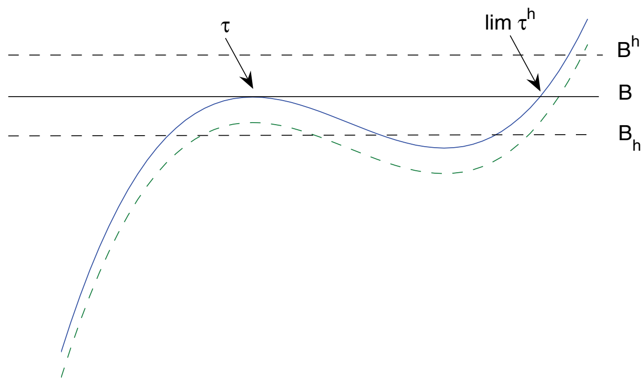

FIG. 5.1. *Tangency problem.*

By virtue of our convergence rate result from the previous section, we conclude that the tangency problem will not occur in probability and in L^1. To see this, consider the control-independent dynamic system. Assumption (H4) is satisfied in this situation (see the previous section). Therefore, we can conclude $|E\tau^h - E\tau| < Kh^{\frac{1}{2}}$ by Theorem 5.1. In fact, we can obtain a stronger result if we carefully examine the proof of Theorem 4.2. In this case, $f \equiv 1$ and $\rho = 1$. To get the upper bound of $(\tau^h - \tau)^+$, we follow the procedure of step 3 in Theorem 4.2; we can obtain the estimates for I_1, I_3, I_4, and I_5 as previously. Note that $I_2 = 0$. We thus conclude $E(\tau^h - \tau)^+ \leq Kh$. Similarly, we obtain $E(\tau^h - \tau)^- \leq Kh^{1/2}$ by step 2. So, we have $E|\tau^h - \tau| \leq Kh^{1/2}$, which implies $\tau^h \to \tau$ in L^1. Then Chebyshev's inequality implies that $\tau^h \to \tau$ with probability as desired.

Appendix. Auxiliary results. This appendix contains a number of technical results and complements. They are needed and used in the proof of the rates of convergence.

LEMMA A.1. *Suppose that* $\phi(x)$ *satisfies*

$$\phi(x) = k(x) + p(x)\phi(x+h) + q(x)\phi(x-h) \quad \forall x \in (-B, B)_h, \tag{A.1}$$

with boundary conditions

$$\phi(x) = u \quad \forall x \geq B, \quad \phi(x) = v \quad \forall x \leq -B,$$

where $(-B, B)_h$ *is as defined in* (2.5). *Suppose that* $k(\cdot)$ *is a real-valued function satisfying* $|k(x)| < Kh^{\gamma_0}$ *for some* $\gamma_0 \geq 2$, *and that there is a* $\delta > 0$ *such that*

$$\delta \leq \frac{1}{2} - Kh \leq p(x) \leq \frac{1}{2} + Kh \leq 1 - \delta, \quad q(x) + p(x) = 1.$$

Then for all $x, y \in [-B, B]_h$,

$$|\phi(x) - \phi(y)| \leq \bar{K}h^{\gamma_0 - 2}|x - y|. \tag{A.2}$$

In particular, when $\phi(\pm B) = 0$, *we have*

$$0 \leq |\phi(x)| \leq \bar{K}h^{\gamma_0-2}(|B - x| \wedge |B + x|) + O(h^{\gamma_0-1}), \tag{A.3}$$

where $\bar{K} = e^{8BK}$.

Proof. Note that (A.1) is a dynamic programming equation of the cost function for a certain Markov chain. Suppose ϕ is a solution of (A.1) with running cost $k(x)$, and $\bar{\phi}$ is a solution of (A.1) with running cost $|k(x)|$. By linearity, $-\bar{\phi}$ is a solution of (A.1) with running cost $-|k(x)|$. Note that $-|k(x)| \leq k(x) \leq |k(x)|$ for all x. By a comparison result with the running cost, one can show $-\bar{\phi} \leq \phi \leq \bar{\phi}$ for all x. Thus, to estimate $|\phi(x)|$, we can simply assume that $k(x)$ is nonnegative for all $x \in (-B, B)_h$.

Let $\hat{x} \in (-B, B)_h$ be an extreme point in the sense

$$\phi(\hat{x}) = \sup_{x \in (-B,B)_h} \phi(x). \tag{A.4}$$

Hence, we have $\phi(\hat{x}) \geq \phi(\hat{x} + h)$. Using $\hat{x} + h$ in (A.1),

$$p(\hat{x} + h)(\phi(\hat{x} + h) - \phi(\hat{x} + 2h)) = k(\hat{x} + h) + q(\hat{x} + h)(\phi(\hat{x}) - \phi(\hat{x} + h)) \geq 0. \tag{A.5}$$

This implies

$$\phi(\hat{x} + h) \geq \phi(\hat{x} + 2h). \tag{A.6}$$

An induction argument yields

$$\phi(\hat{x}) \geq \phi(\hat{x} + h) \geq \phi(\hat{x} + 2h) \geq \cdots \geq \phi(B) = u, \tag{A.7}$$

and similarly

$$\phi(\hat{x}) \geq \phi(\hat{x} - h) \geq \phi(\hat{x} - 2h) \geq \cdots \geq \phi(-B) = v. \tag{A.8}$$

Next, we examine the difference $\phi(\hat{x}+(i-1)h)-\phi(\hat{x}+ih)$ for each positive integer i. First, this difference is nonnegative. Using $\hat{x}$ in (A.1) yields

$$p(\hat{x})(\phi(\hat{x}) - \phi(\hat{x} + h)) + q(\hat{x})(\phi(\hat{x}) - \phi(\hat{x} - h)) = k(\hat{x}) \leq Kh^{\gamma_0}. \tag{A.9}$$

Note that both terms on the left-hand side of (A.9) are nonnegative, so

$$\phi(\hat{x}) - \phi(\hat{x}+h) \le 2Kh^{\gamma_0} + O(h^{\gamma_0+1}). \tag{A.10}$$

For any positive integer i, applying $\hat{x}+ih$ to (A.1),

$$\begin{aligned} &p(\hat{x}+ih)(\phi(\hat{x}+ih) - \phi(\hat{x}+(i+1)h)) \\ &\quad \le q(\hat{x}+ih)(\phi(\hat{x}+(i-1)h) - \phi(\hat{x}+ih)) + Kh^{\gamma_0}. \end{aligned} \tag{A.11}$$

Dividing both sides of (A.11) by $p(\hat{x}+ih)$,

$$\begin{aligned} &\phi(\hat{x}+ih) - \phi(\hat{x}+(i+1)h) \\ &\le (1+4Kh)(\phi(\hat{x}+(i-1)h) - \phi(\hat{x}+ih)) + 2Kh^{\gamma_0}(1+2Kh). \end{aligned} \tag{A.12}$$

Equation (A.12) can be written recursively as

$$\begin{aligned} &\phi(\hat{x}+ih) - \phi(\hat{x}+(i+1)h) + \frac{1}{2}h^{\gamma_0-1}(1+2Kh) \\ &\quad \le (1+4Kh)\left(\phi(\hat{x}+(i-1)h) - \phi(\hat{x}+ih) + \frac{1}{2}h^{\gamma_0-1}(1+2Kh)\right). \end{aligned} \tag{A.13}$$

It follows that

$$\begin{aligned} &\phi(\hat{x}+ih) - \phi(\hat{x}+(i+1)h) + \frac{1}{2}h^{\gamma_0-1}(1+2Kh) \\ &\le (1+4Kh)^i\left(\phi(\hat{x}) - \phi(\hat{x}+h) + \frac{1}{2}h^{\gamma_0-1}(1+2Kh)\right) \\ &\le (1+4Kh)^{2B/h}\left(\frac{1}{2}h^{\gamma_0-1} + 3Kh^{\gamma_0} + O(h^{\gamma_0+1})\right) \\ &\le \bar{K}h^{\gamma_0-1} + O(h^{\gamma_0}) \quad (\text{where } \bar{K} \triangleq e^{8BK}). \end{aligned} \tag{A.14}$$

Therefore,

$$\phi(\hat{x}+ih) - \phi(\hat{x}+(i+1)h) \le \bar{K}h^{\gamma_0-1},$$

and similarly

$$\phi(\hat{x}-ih) - \phi(\hat{x}-(i+1)h) \le \bar{K}h^{\gamma_0-1},$$

which implies

$$|\phi(x) - \phi(x+h)| \le \bar{K}h^{\gamma_0-1} \quad \forall x \in [-B,B)_h.$$

Without loss of generality, assume $x < y$,

$$\begin{aligned} |\phi(x) - \phi(y)| \le \sum_{z\in[x,y)_h} |\phi(z) - \phi(z+h)| &= \bar{K}h^{\gamma_0-1}\left(\frac{y-x}{h} + 1\right) \\ &= \bar{K}h^{\gamma_0-2}(y-x) + O(h^{\gamma_0-1}). \end{aligned} \tag{A.15}$$

This implies the lemma. ∎

THEOREM A.2. *Let $V_i^B(x)$ be the solution of (2.4). Then $V_i^B(\pm B) = 0$ for all $B \in G$, and $V_i^B(x)$ is Lipschitz in x and B. That is,*

$$|V_i^{B_1}(x_1) - V_i^{B_2}(x_2)| \le K(|B_1 - B_2| + |x_1 - x_2|) \quad \forall i \in \mathcal{M}, (B_1,x_1),(B_2,x_2) \in G^2.$$

This result continues to hold for the value function (2.4) *when* inf *is replaced by* sup.

Proof. Using (H1)–(H3), the system of HJB equations (2.4) admits a unique viscosity solution (see [22, Theorem A.24]). Hence, by (2.4), $V_i^B(x)$ for any given i is the solution of

$$\inf_{r\in U}\{\tilde{L}^r\varphi(x)+\tilde{f}(x,r)\}=0,\quad \varphi(\pm B)=0, \tag{A.16}$$

where

$$\begin{aligned}\tilde{L}^r\varphi(x)&=\frac{1}{2}\sigma_i^2(x)\frac{d^2\varphi(x)}{dx^2}+b_i(x,r)\frac{d\varphi(x)}{dx}+q_{ii}\varphi(x),\\ \tilde{f}(x,r)&=f_i(x,r)+\sum_{j\neq i}q_{ij}V_j^B(x).\end{aligned}$$

By [12, Theorem 5, p. 24], (A.16) admits a smooth solution $\varphi\in C^2(G)$. Therefore, (2.4) has a unique smooth solution $V_i^B(\cdot)$ for each $i\in\mathcal{M}$, and $V_i^B(x)$ is Lipschitz continuous in x on compact set G. In particular, since $V_i^B(\pm B)=0$,

$$|V_i^B(x)|=|V_i^B(x)-V_i^B(B)|\le K|x-B|$$

and

$$|V_i^B(x)|=|V_i^B(x)-V_i^B(-B)|\le K|x+B|.$$

Therefore, we have

$$|V_i^B(x)|\le K(|x-B|\wedge|x+B|). \tag{A.17}$$

For a given initial state $(i,x)\in\mathcal{M}\times(-B_1,B_1)$ and $0<B_1<B_2$, let $\bar{m}$ be an optimal relaxed control. For convenience, set $\tau\triangleq\tau_{B_1}^{i,x,\bar{m}}$. By the dynamic programming principle,

$$\begin{aligned}V_i^{B_2}(x)&\le V_i^{B_1}(x)+E[V_{\alpha_\tau}^{B_2}(X_\tau^{i,x,\bar{m}})]\\ &\le V_i^{B_1}(x)+K|B_2-B_1|,\\ &\qquad\text{since } X_\tau^{i,x,\bar{m}}=\pm B_1 \text{ and by (A.17)}.\end{aligned}$$

Similarly,

$$\begin{aligned}V_i^{B_2}(x)&=J_i^{B_1}(x,\bar{m})+E[V_{\alpha_\tau}^{B_2}(X_\tau^{i,x,\bar{m}})]\\ &\ge V_i^{B_1}(x)-K|B_2-B_1|.\end{aligned}$$

This completes the proof. ∎

COROLLARY A.3. *For given $m\in\Gamma$, we have*

$$|J_i^{B_1}(x,m)-J_i^{B_2}(x,m)|\le K|B_1-B_2|\quad\forall(i,x)\in\mathcal{M}\times G.$$

Proof. Without loss of generality, let $0<B_1<B_2$. Observe that

$$J_i^{B_2}(x,m)-J_i^{B_1}(x,m)=E[J_{\alpha_\tau}^{B_2}(\pm B_1,m)],$$

where $\tau=\inf\{t:X_t\notin(-B_1,B_1)\}$.

From Theorem A.2,

$$V_i^B(x) = \inf_m J_i^B(x,m) \geq -K(|x-B| \wedge |x+B|).$$

Similarly, we have

$$W_i^B(x) \triangleq \sup_m J_i^B(x,m) \leq K(|x-B| \wedge |x+B|).$$

Since

$$-K|B_1 - B_2| \leq V_{\alpha_\tau}^{B_2}(\pm B_1) \leq J_{\alpha_\tau}^{B_2}(\pm B_1, m) \leq W_{\alpha_\tau}^{B_2}(\pm B_1) \leq K|B_1 - B_2|,$$

the desired result follows. □

LEMMA A.4. *Let $X_t = bt + \sigma W_t$, and let $\tau = \inf\{t : |X_t| > h\}$. Then*

$$P(X_\tau = \pm h) = \frac{1}{2} \pm \frac{bh}{2\sigma^2} + O(h^3),$$
$$E\tau = \frac{h^2}{\sigma^2} + O(h^4),$$
$$E\tau^2 = \frac{h^4}{\sigma^4} + O(h^6).$$

Proof. For the first identity, we can solve for the explicit solution of the related PDE and get

$$P(X_\tau = \pm h) = \frac{\pm \exp(\pm 2bh/\sigma^2) \mp 1}{\exp(2bh/\sigma^2) - \exp(-2bh/\sigma^2)}.$$

The first and second moments of τ can be obtained from the moment generating function given in [11, p. 100]. □

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