

```

        ENDIF
        CBY(K) = CYY
55      CONTINUE
        CDY(0) = -CBY(1)
        DO 60 K=1, NM                <<< Calculate the derivative of  $Y_n$ 
60      CDY(K) = CBY(K-1) - K/Z * CBY(K)
        RETURN
    END

```

5.5 COMPUTATION OF $J_\nu(z)$ AND $Y_\nu(z)$ WITH AN ARBITRARY ORDER

In this section we discuss the computation of the Bessel functions of an arbitrary order and present two computer programs: one for real arguments and the other for complex arguments. The Bessel functions of an arbitrary order can still be computed using the recurrence relation (5.1.21) with the starting values evaluated from a series expansion for small arguments and an asymptotic expansion for large arguments. To be more specific, given an order ν , we can first find ν_0 such that $\nu = \nu_0 + n$, where $0 \leq \nu_0 < 1$ and n is an integer. Then for small arguments, we can evaluate $J_{\nu_0}(z)$, $J_{\nu_0+1}(z)$, $Y_{\nu_0}(z)$, and $Y_{\nu_0+1}(z)$ using the series expansions given by (5.1.2)–(5.1.4), which can be rewritten in a form more suitable for numerical computation as

$$J_{\pm\nu}(z) = \frac{(z/2)^{\pm\nu}}{\Gamma(1 \pm \nu)} \left\{ 1 + \sum_{m=1}^{\infty} \prod_{k=1}^m \left[-\frac{(z/2)^2}{k(k \pm \nu)} \right] \right\} \quad (5.5.1)$$

For large arguments, we can evaluate $J_{\nu_0}(z)$, $J_{\nu_0+1}(z)$, $Y_{\nu_0}(z)$, and $Y_{\nu_0+1}(z)$ using the asymptotic expansions given by (5.2.5) and (5.2.6), with (5.2.7) and (5.2.8) being evaluated directly since ν is now an arbitrary number. For the convenience of numerical computation, (5.2.7) and (5.2.8) can be rewritten as

$$P(\nu, z) \sim 1 + \sum_{k=1}^{M_k} \prod_{j=1}^k \left\{ -\frac{[\mu - (4j-3)^2][\mu - (4j-1)^2]}{2j(2j-1)(8z)^2} \right\} \quad (5.5.2)$$

$$Q(\nu, z) \sim \frac{\mu-1}{8z} \left(1 + \sum_{k=1}^{M_k} \prod_{j=1}^k \left\{ -\frac{[\mu - (4j-1)^2][\mu - (4j+1)^2]}{2j(2j+1)(8z)^2} \right\} \right) \quad (5.5.3)$$

Once $J_{\nu_0}(z)$, $J_{\nu_0+1}(z)$, $Y_{\nu_0}(z)$, and $Y_{\nu_0+1}(z)$ are evaluated, we can compute $J_\nu(z)$ by backward recurrence and $Y_\nu(z)$ by forward recurrence for a real z .

The following computer program implements the algorithm described above and evaluates the Bessel functions of an arbitrary order with real arguments