

Stable Group Purchasing Organizations

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Abstract

In this paper, we study the stability of Group Purchasing Organizations (GPOs). GPOs exist in several sectors and benefit its members through quantity discounts and negotiation power when dealing with suppliers. However, despite several obvious benefits, GPOs suffer from member dissatisfaction due to unfair allocations of the accrued savings among its members. We first explore the benefits of allocation rules that are commonly reported as being used in practice. We characterize stable coalitional outcomes when these rules are used and provide conditions under which the grand coalition emerges as a tenable outcome. These conditions are somewhat restrictive. We then propose an allocation mechanism based on the marginal value of a member's contribution and find that this leads to stable GPOs in many scenarios of interest. In this analysis, we look at discount schedules that encompass a large class of practical schedules and analyze cases when purchasing requirements of the members are both exogenous as well as endogenous. We use a concept of stability that allows for players to be farsighted, i.e., players will consider the possibility that once they act (say by causing a defection), another coalition may react, and a third coalition might in turn react, and so on, nullifying their original advantage in making the initial move.

1. Introduction

Group purchasing organizations (GPOs) are coalitions of several firms (buyers) who pool their purchasing requirements and buy large quantities of a particular product from a seller. The advantage of belonging to a GPO is evident. GPOs are able to take advantage of significant quantity discounts from the seller and transaction costs may be lowered by bundling different orders. In certain sectors, GPOs have high negotiation power and receive preferred terms of trade.

GPOs, henceforth also referred to as purchasing coalitions, are seen in various industry sectors. The origins of such coalitions can be traced back to the evolution of cooperatives, where cooperatives acted as purchasing and selling coalitions. Among non-commodity industries, health care was one of the earliest to see the formation of large GPOs. The first healthcare GPO was created in 1910 by the Hospital Bureau of New York. Today, purchasing coalitions can be seen in virtually every sector spanning health care (Doucette 1997, Mitchell 2002), education (Doucette 1997), retail (Zentes and Swoboda 2000, Dana 2006), etc. A study by Hendrick (1997) shows that in the U.S. about 20% of firms belong to some purchasing consortia, and this trend is rapidly increasing (Major 1997). Today, 97% of all not-for-profit, non-governmental hospitals in the United States participate in some form of group purchasing, with Provista, Novation, and Innovatix being some leading purchasing consortia. Examples of other large GPOs are FoodBuy in the grocery industry and PrimeAdvantage, a large industrial manufacturers' GPO.

The advent of electronic commerce has added to this trend internationally. Currently, there are several avenues on the web where individual buyers can virtually aggregate orders to avail discounts (known as group buying; see Anand and Aron 2003). Mercata is one such example in the United States; a related and extremely popular shopping strategy in China is Tuangou, which is attracting considerable attention.

Despite the touted benefits of forming such organizations, the business and academic literature lament the fact that GPOs do not sustain themselves, often due to disagreements between members. GPOs see membership numbers often fluctuate. Further, problems often arise due to unequal member contributions, with powerful members building barriers to keep out smaller participants from entering the GPO (Bloch et al. 2008). The very nature of these coalitions causes reasons for conflict. Consider a group of buyers, each with a certain purchasing requirement, who decide to form a coalition. The discount from the seller is based on the aggregated purchasing quantity. Thus, each member of the coalition is able to receive a lower price than what he would have received based solely on his individual order. However, when a set of buyers with heterogeneous

requirements form a coalition, it is not immediately clear how the benefits of this lowered price are to be allocated among coalition members. This discord leads to a lack of commitment among coalition members and to potential instability (Heijboer 2003). A comprehensive study of several purchasing consortia in Europe indicates that over a quarter of such coalitions are acutely aware of inherent unfairness in splitting the savings obtained by the coalition. An often-used mechanism that allocates savings internally to the coalition, known as equal price (i.e., one where all members of the coalition pay the same price per item), is an excellent example that opens itself to this criticism. The common wisdom that gains accrued by the coalitions may far surpass any cause for discord is increasingly questioned by coalition members and this may explain the short life of several such consortia (Aylesworth 2003).

Thus, an important strategic issue to be considered when setting up a GPO seems to be one of “fair allocations” of the gains realized by such a coalition. Evidently, the issue of allocations of gains is intimately tied to the eventual stability of these coalitions. Despite recognizing the importance of this topic, neither the literatures in operations management that deals with purchasing nor the ones in economics analyze a comprehensive model of purchasing consortia and offer any robust remedies.

In this paper, we examine how allocations among group members in a purchasing coalition should be designed (especially when contributions by individual members may not be equal) and analyze the related issues of stability of purchasing coalitions. Our analysis uses the theory of cooperative games. This choice is natural, given that an important aspect of our problem is that of finding fair allocations. Our analysis takes the following path. We first search the extant literature (both academic and industry) to find a good representation of a quantity discount schedule. The functional form we end up using fits best with what is seen in practice and possesses a few attractive analytical properties as well. Schotanus (2007) analyzes quantity discount schedules that are often used in group buying settings and uses this data to fit a function that best describes such contracts. We use this as a building block in our paper. In terms of dividing the gains of a coalition, we use and suggest a few allocation rules that make sense from both a practical perspective as well as are related to theoretical concepts from cooperative games. We first look at the two allocation rules most commonly used in practice. These are (i) the equal allocation principle, which allocates to each member the same portion of accrued savings; and (ii) the quantity-based proportional allocation rule, which allocates the savings proportionally to the buyers’ ordering quantities. We show that these rules are often problematic and inherently cause member dissatisfaction that usually lead to defections by members. With the aim of sustaining GPOs, we propose the Shapley value (Shapley

1953), which tries to compute the marginal contribution of an individual member of the coalition as an allocation rule. We show that using the marginal contribution leads to a greater set of instances in which the GPO is stable. All of these rules are well understood and have theoretically attractive properties. In a later section, we define, analyze and comment on each one of these and the appropriateness of their usage. The issue of allocations, as mentioned earlier, is tied to the question of stability of the alliance. We use cooperative game theory to analyze the stability of coalitions. A commonly used concept of stability, popular in the operations literature, is the core (Gillies 1959). The core distinguishes allocation rules that yield a stable alliance of all players (roughly speaking, no set of players have an immediate incentive to defect from their joint alliance when an allocation is in the core), but suffers from the problem of myopia. That is, it precludes the possibility that players and sub-coalitions may consider the possibility that once they act (say, by causing a defection), another coalition may react, and a third coalition might in turn react, and so on, nullifying their original advantage in making the initial move. A concept of stability that takes such a farsighted view of players is the Largest Consistent Set (LCS, Chwe 1994). In our analysis, we allow for players to be farsighted and thus primarily use the LCS to evaluate the stability of coalitions.

Our main findings when considering the aforementioned allocation rules are as follows. We first look at a scenario in which the requirement of each firm (i.e., its order quantity) is exogenous to the discussion and thus independent of the specifics of the discount schedule. Such a scenario is realistic in several public-sector GPOs (the healthcare sector which features some of the largest GPOs in today's economy being a good example). Here, purchase orders are periodically determined based on needs, and less on other factors such as profit maximization or re-selling decisions. In this setting, when buyers are homogeneous, the alliance of all buyers (the grand coalition) is stable and sustains itself independent of the allocation rules. This result is not surprising. However, when the buyers are heterogeneous, the Shapley value alone produces fair allocations that ensure the stability of the grand coalition. When either equal allocation rules or quantity-based proportional allocation rules are used and one looks at the farsighted stability of coalitions, there is a strong tendency in which the buyers with large orders split to form their own GPO, leaving buyers with smaller contributions to themselves. This leads to the following insight—if one needs to sustain a GPO with heterogeneous buyers, contracts that allocate savings between buyers need to be carefully arrived at, using, for instance, the Shapley value allocations. That is, allocation rules that are drawn, before membership contracts are written up, need to do a calculation that takes into account the marginal contribution of members of the coalition. Several of the currently followed allocation

rules that fail to do this will result in eventual instability, even if they seem fair (for instance, quantity-based proportional allocation) at the outset.

Next, we look at a scenario in which the quantity required by each buyer is endogenous to the model. An illustrative case is that of a profit-maximizing price-setting firm which decides how much its requirements are by considering both the quantity discount schedule offered by the seller and its own downstream price-driven demand. The firm under question simply sets its demand curve and the discount schedule (supply curve) equal to each other and derives its requirements based on this intersection. As one can imagine, this analysis can be algebraically tedious, as solving for the order quantity may not be simple. When we consider *linear* demand and discount schedules, we are able to show that several of our results and insights from previous analysis with exogenous demands still hold. In particular, we show that it is still advisable to use a Shapley-value-based allocation to guarantee the stability of the GPO. We also show that when quantity decisions are endogenous and one uses either the equal allocation or the quantity based allocation rules, under certain assumptions our earlier result continues to hold (i.e., large coalitions coalesce together and leave the smaller members out). This prediction of ours is seen widely in practice and is also the topic of several studies and reports on purchasing organizations (see, for instance, Bloch et al. 2008).

In summary, our central message in this paper is that using the Shapley value to allocate savings to the individual members of a GPO is essentially robust to stability concepts, heterogeneity of buyer requirements, and the perception of fairness among buyers, and thus yields the best chance to sustain a GPO.

Using numerical examples we show that, in general, when one looks at endogenous setting, the games under question lose several of the nice properties that facilitate an elegant analysis. Games such as these may neither be convex nor concave. The implication is that, in the presence of non-linear schedules and demands (for instance, think of a demand with constant elasticity), or in instances where the buyers face significant internal decisions such as the retail price of their product, one needs to be very careful in prescribing allocation rules. In general, allocation rules required to preserve the alliance may be far too complicated to implement, and in some instances may not even exist. Thus, in such instances, it may well be in the health of the GPO to negotiate less complicated discount schedules that may lead to slightly lower overall savings.

We believe our findings above yield important insights to firms that contemplate joining purchasing coalitions, or to intermediaries who wish to create successful and efficient GPOs. Since the creation of GPOs has substantial consequences to social surplus and creates efficiencies in many

supply chains, we believe it is particularly important for the relevant players to understand and pursue strategies that will contribute to its success.

The rest of the paper is organized as follows. In Section 2, we study the model in which the purchasing requirements of the individual buyers are exogenous, introduce the three allocation rules considered in this paper, and analyze their properties. In Section 3, we introduce the farsighted stability concept (the largest consistent set), which we then apply to our model to obtain farsighted-stable outcomes in Section 4. We analyze the model in which the purchasing requirements are price-dependent under a linear discount scheme in Section 5, while in Section 6 we discuss some significant extensions of this model which allows for a fuller analysis of situations when requirements are endogenous. Finally, we conclude in Section 7 and provide some directions for future work.

2. Model with Exogenous Quantities

We begin our analysis in this section by describing the basic model that looks at the creation of a GPO. Suppose that n buyers decide to form a purchasing consortium in order to benefit from available quantity discounts or economies of scale that generate lower per unit prices. There are several reasons for buyers to unite and form such a coalition. In this paper, we isolate and focus exclusively on the effect of price discounts. That is, in our setting, there is a seller who announces a discount schedule that buyers can avail of. Thus, we assume that unit purchase price per item, p , is a function of quantity purchased, q , and that the function $p(q)$ reflects a discount (i.e., is decreasing, continuous, and convex in q). In practice, suppliers may sometimes announce discount schedules that are characterized by discrete step sized schedules. The appropriateness of continuous quantity discounts is well known and used widely in the operations literature (see, for instance, Mitchell 2002). Schotanus (2007) argues, based on data, that a function of type

$$p(q) = \alpha + \beta q^{-\eta}, q \geq 1, \alpha > 0 \tag{1}$$

fits well with different types of quantity discounts. He analyzes 66 discount schedules and shows few discrepancies (in three cases, due to outlying points). The form of the discounting scheme (1) is rather general, but imposes some restrictions on its parameters. When $\eta > 0$, the discount function has a positive steepness and we require that $\beta > 0$; when $\eta < 0$, the discount function has a negative steepness and thus one requires that $\beta < 0, -1 \leq \eta < 0$. We impose an additional requirement—that the amount transferred to the seller, $qp(q)$, is a concave increasing function. This assumption seems to hold in most practical schedules identified in the literature. Note that a

linear discount scheme, $p(q) = \bar{\alpha} - \bar{\beta}q$, which will be used later in this paper, is a special case of (1) obtained when $\eta = -1$ and $\beta < 0$. The above form of a discount schedule seems especially useful when the number of buyers is large. Thus, for the remainder of this paper, we will continue to assume a continuous quantity discount function with the understanding that in several realistic scenarios (when n is large or when piecewise linear discounts are used) our approximation is reasonable.

We first begin by analyzing a setting in which the purchasing requirements of the individual buyers are exogenous to the model. That is, buyer i requires quantity q_i , and this quantity decision is not driven by the discount schedule available to him. In the literature on purchasing coalitions, this seems to be a common assumption. In a later section, we relax this assumption and assume that each buyer faces a demand curve and makes his purchasing decision based on the discount schedule (supply curve) and his demand curve. Continuing with this exogenous assumption, we let buyer i to require q_i , and we denote $\mathbf{q} = (q_1, q_2, \dots, q_n)$. The total order quantity of the GPO with n buyers in the alliance is then $\sum_{i=1}^n q_i = Q$. The rationale for forming the GPO is clear even in this simple setting. The per unit price paid by the coalition is $p(Q)$, which is smaller than $p(q_i)$, the per unit price that buyer i would get by transacting with the seller on his own. It is not uncommon that in such consortia each member pays the same per unit price, $p(Q)$, and hence this model seems to be often adopted in research papers in collaborative purchasing (Chen and Roma 2008, Chen and Yin 2008, Keskinocak and Savaseneril 2008). In this case, the savings observed by buyer i due to alliance membership can be written as $q_i[p(q_i) - p(Q)]$. Let us denote by N the set of all buyers, i.e., $N = \{1, 2, \dots, n\}$. Then, using terminology and notations from cooperative games, the *value* of the *grand coalition*, $v(N)$, is simply the total savings generated by the entire consortium, and is given by

$$v(N) = \sum_{j=1}^n q_j p(q_j) - Q p(Q). \quad (2)$$

Note that the purchasing requirements of the buyers are not necessarily equal. Thus, buyers may contribute unequal amounts to the total quantity, Q . A direct consequence is that some of the alliance members may justifiably feel that their share of the savings does not reflect the magnitude of their contribution towards the savings generated by the alliance. To quantify this sentiment, we can write the contribution of buyer i to the alliance as

$$q_i p(q_i) + p(Q - q_i)(Q - q_i) - Q p(Q),$$

which after rearranging terms can be written as

$$q_i [p(q_i) - p(Q)] + (Q - q_i) [p(Q - q_i) - p(Q)].$$

We can interpret the above expression as follows. Any buyer, i , besides observing savings of $q_i[p(q_i) - p(Q)]$ himself, which are generated by both himself and the remaining buyers, also contributes a saving of $(Q - q_i)[p(Q - q_i) - p(Q)]$ for the remaining alliance members. This interpretation allows us to easily generate examples wherein a buyer who contributes more receives lower overall savings than a buyer that contributes less to the value of Q . This phenomena motivates the discussion that, in order to sustain a successful consortium, one may want to propose some alternative methods for allocation of savings among the alliance members.

In the rest of this section, we put forward a framework that allows us to analyze the central question of interest; that is, what kind of allocation mechanisms should be employed to ensure the success of the GPO? We need to ensure that no sub-coalition of buyers, $S \subset N$, wants to leave the GPO (the coalition N) and form their own GPO. Buyers may decide to leave for a myriad of reasons; our focus is on two of them. First, the buyers may believe that, by defecting, they can create and allocate the resulting savings more efficiently than the original GPO. Second (which is closely related to the first reason), the buyers may feel that individual gains from being in the GPO are allocated "unfairly" with respect to their contribution. The second reason can sometimes be hard to isolate. For instance, a group may defect and even suffer some loss if there is perceived unfairness due to, perhaps, a free rider in the coalition. To isolate these effects in the analysis, we introduce some additional notation.

Applying the concepts from game theory, we will define a savings game as follows: for any coalition, $S \subseteq N$, we denote by $v(S)$ value generated by that coalition. In this part of our analysis, we identify the value of coalition S with the total savings generated by joint order-placing of buyers in S . Then, $v(S)$ can be written as

$$v(S) = \sum_{j \in S} q_j \left[p(q_j) - p \left(\sum_{j \in S} q_j \right) \right].$$

We further denote by $\varphi_i(v) \in \mathbb{R}$ allocation of total savings, $v(N)$, received by buyer i . When it is clear from the context what value function we are referring to, we write φ_i instead of $\varphi_i(v)$, and we denote by $\varphi \in \mathbb{R}^n$ the allocation vector.

We now describe the logic behind the allocation rules we analyze in this paper. From a theoretical perspective, allocation rules are often tested based on the number of attractive properties they satisfy from a fairly standard list of requirements (see Myerson 1997). This list includes several reasonable properties such as symmetry, efficiency, additivity, individual rationality, etc. Of the many popular allocation rules from theory, such as the Shapley value (Shapley 1953), the nucleolus

(Schmeidler 1969) and the compromise value (Driessen 1985), the Shapley value turns out to be the most attractive for the savings game. Moreover, recall that we assumed $p(q)$ is a convex function and $qp(q)$ is concave and increasing in q ¹. This seems to be a reasonable assumption that holds for most schedules in practice. In particular, the total payments made to the seller, as one would expect, increases as the quantity purchased becomes larger, and this increase sees a diminishing return. When this property is factored in, the savings game turns out to be convex; that is,

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S)$$

whenever $S \subset T \subseteq N \setminus \{i\}$. Convexity implies the superadditivity of the game; that is, when any two disjunct coalitions join, the total savings generated by their members increase. A direct implication of this fact is that the Shapley value satisfies a myopic stability property (i.e., the Shapley value belongs to the core when the game is convex, and no subset of players wants to defect from N). In summary, for our savings game, the Shapley value is the only allocation rule that satisfies all the common requirements recognized in the literature. This strongly suggests that, among the theoretical rules, Shapley value may be the best suited for our purposes.

From a practical perspective, the two most commonly used allocation rules in group purchasing are equal allocations (all players receive equal share of savings) and the quantity-based proportional rule (players receiving shares proportional to quantities ordered). We believe that the reason for this is the apparent simplicity of these rules. Other rules (that are rarely mentioned in the business press) such as savings-based proportional rule (players receiving shares proportional to percentage of savings) give rise to obvious unfair situations, such as the buyer contributing the most to the purchasing quantity can receive the smallest portion of savings. Therefore, we do not analyze these rules.

In summary, in the remainder of the paper, we will concentrate on three allocation rules: the Shapley value, quantity-based proportional allocation, and equal allocations. We want to compare the effects of applying these different allocation rules, to investigate when the joint purchasing alliance of all members is a stable outcome, as well as what would be the resulting stable structures if the grand purchasing organization is unstable. As mentioned earlier, the notion of stability we use (the largest consistent set, LCS) allows for players to be farsighted. We describe this in detail in a later section. We now turn our attention to describing the three allocation rules that we introduced above and use in the rest of the paper.

¹Our discount function (1) satisfies this assumption when $0 < \eta < 1$, or when $\eta = 1$ for $q \leq \alpha/(2\beta)$.

2.1 Equal Allocations

The simplest allocation of savings would be to give an equal portion to each buyer,

$$\varphi_i^E = \frac{\sum_{j=1}^n q_j p(q_j) - Qp(Q)}{n},$$

so that buyer i is paying

$$\begin{aligned} c^E(\mathbf{q}) &= q_i p(q_i) - \frac{\sum_{j=1}^n q_j p(q_j) - Qp(Q)}{n} \\ &= \frac{\sum_{j=1}^n [q_i p(q_i) + q_i p(Q) - q_i p(Q) - q_j p(q_j) + q_j p(Q)]}{n} \\ &= \frac{\sum_{j \neq i} \{ [q_i p(q_i) - q_i p(Q)] - [q_j p(q_j) - q_j p(Q)] \}}{n} + q_i p(Q). \end{aligned}$$

Thus, the buyer who contributes the most can pay more or less than $q_i p(Q)$, depending on the range of the quantities ordered by all buyers (see Example 1).

2.2 Quantity-Based Proportional Allocation Rule

Another simple way of allocating savings would be to distribute them in proportion with the contribution of different buyers,

$$\varphi_i^P = \frac{q_i \left[\sum_{j=1}^n q_j p(q_j) - Qp(Q) \right]}{Q},$$

so that buyer i is paying

$$\begin{aligned} c^P(\mathbf{q}) &= q_i p(q_i) - \frac{q_i \left[\sum_{j=1}^n q_j p(q_j) - Qp(Q) \right]}{Q} \\ &= \frac{q_i \left[Qp(q_i) - \sum_{j=1}^n q_j p(q_j) + Qp(Q) \right]}{Q} \\ &= \frac{q_i}{Q} \sum_{j=1}^n q_j [p(q_i) - p(q_j) + p(Q)] \\ &= q_i p(Q) + \frac{q_i}{Q} \sum_{j=1}^n q_j [p(q_i) - p(q_j)] \end{aligned}$$

In this case, a buyer who contributes the most pays less than $q_i p(Q)$, while a buyer who contributes the least pays more than $q_i p(Q)$.

2.3 Shapley Value Allocations

Another possibility is to distribute the savings according to the Shapley value allocations. Consider all possible orderings of players, and define a marginal contribution of player i with respect to a

given ordering as his marginal worth to the coalition formed by the players before him in the order, $v(\{1, 2, \dots, i-1, i\}) - v(\{1, 2, \dots, i-1\})$, where $1, 2, \dots, i-1$ are the players preceding i in the given ordering. Shapley value is obtained by averaging the marginal contributions for all possible orderings. If we denote by $|S|$ number of buyers in alliance S , Shapley allocation to player i can then be written as

$$\varphi_i^N(v) = \sum_{\{S:i \in S\}} \frac{(|S|-1)!(n-|S|)!}{n!} (v(S) - v(S \setminus \{i\})).$$

Chen and Yin (2008) use Shapley value allocations in their paper; however, their game differs from ours, as the value of an coalition in their article is the negative of the coalition's total purchasing cost (hence, they do not analyze the savings game).

2.4 Comparisons

We next provide an example to illustrate differences among the allocation rules introduced above.

EXAMPLE 1. Suppose that $p(q) = 10 + 30q^{-0.5}$, and that $N = \{1, 2, 3\}$. Ordering quantities, q_i , unit prices, $p(q_i)$, and costs before cooperation, $q_i p(q_i)$, together with costs, C_i , and savings allocation, φ_i , under different allocation rules are given in Table 1. Superscripts 0, E , P , and S denote equal price, equal allocations, proportional rule, and Shapley value, respectively. Note that when the buyers form a purchasing alliance and jointly buy the product, $Q = 150$ and $p(Q) = 12.45$.

i	q_i	$p(q_i)$	$q_i p(q_i)$	$q_i p(Q)$	C_i^E	C_i^P	C_i^S	φ_i^0	φ_i^E	φ_i^P	φ_i^S
1	20	16.7	334	249	257	303	266	85	77	31	68
2	30	15.5	464	374	387	418	388	90	77	46	77
3	100	13.0	1,300	1,245	1,223	1,146	1,214	55	77	154	86

Table 1: Ordering quantities, unit prices, cost, and savings under different allocation rules

Thus, with proportional and Shapley allocations, the buyer who contributes the most is allocated the highest share of savings. Note that even this simple analysis seems to indicate that all of the three proposed methods above seem fairer than the often-used scheme in which each buyer pays equal price (see φ_i^0). \square

The next question that needs to be addressed is stability of purchasing alliances—do all alliance members have an incentive to jointly purchase the items, or could there exist a subset of buyers

that benefits from purchasing separately? A popular concept game theoretic of stability used in the operations management literature is the core, which is defined as follows. An allocation φ is a member of the core of if it satisfies

$$\sum_{i \in S} \varphi_i \geq v(S) \quad \forall S \subseteq N,$$

$$\sum_{i=1}^n \varphi_i(v) = v(N).$$

When core allocations are used, no subset of players has an incentive to secede and form its own coalition. The core was introduced to the operations literature in Hartman and Dror (1996) in the newsvendor context and has since then been widely adopted (for example see Hartman et al. 2000, Hartman and Dror 2003, 2005, Chen and Zhang 2008). Thus, in order to induce participation of all buyers in the GPO, one may want to select core allocations. The drawback of the allocations proposed above is that, in general, they do not belong to the core. Consider, for instance, Example 1 above when the proportional allocation rule is applied. If buyers 1 and 2 form a purchasing alliance on their own, they would generate total savings of 86, while under the proportional rule they receive a total of 77. Thus, they would be better off by acting alone. Similarly, suppose that each buyer receives equal share of savings, and that buyer 1 orders 5, buyer 2 orders 90, and buyer 3 orders 100. Then, the total share of savings allocated to players 2 and 3 when all buyers form a consortium equals 52, while by purchasing without buyer 1 they generate savings of 57. Thus, neither of the two practical rules induces all buyers to participate in the consortium, providing that players consider only one-step defections.

The above analysis merits a discussion. The first point to note is that the two practical rules at the outset seem fairer than one in which all buyers pay the same price. However, when one looks at stability concept such as the core, neither of these two rules yield stable allocations. That is, if one were to use these rules, a subset of buyers will defect from the GPO. The second point to note is that when all buyers pay the same price, one can show that the core constraints hold. That is, despite the apparent unfairness, no sub-coalition has an immediate incentive to defect. We need to be careful about interpreting these results. The observation that charging everyone the same price yields a core allocation (i.e., stable outcome) needs to be reconciled with empirical fact that this approach leads to members of the GPO being unsatisfied due to perceived unfairness. Consequently, it appears that among the practical rules, one has to choose between allocation rules that are less “fair” (as they may give the smallest allocations to buyers who contribute the most), but encourage participation of all buyers, and allocation rules that seem more justifiable, but may

incentivize some buyers to defect, thereby destroying the GPO. Now, when we look at the game-theoretic concept in question, recall that, since our game is convex, the Shapley value belongs to the core. The Shapley value, as per its definition, does a calculation that brings to the table the marginal contribution of the players. This proposes a notion of fairness that is easy to describe and explain in practice. This, together with the fact that the Shapley value yields an allocation in the core, leads one to think that perhaps this is the best allocation mechanism to sustain a GPO.

We conclude this section by noting that the concept of stability (i.e., the core) used in the above discussion is static. That is, players are myopic and simply look at one-step deviations by other buyers. We want to investigate whether the same results hold in a dynamic setting, in which buyers are farsighted and consider how others may react to their actions. In order to study this problem we first need to introduce some additional concepts from game theory, which look at stability in a farsighted sense. We do this analysis in the next sections. We continue to focus on the three allocation schemes described above and ignore the equal-price mechanism due to its obvious perceived unfairness.

3. Stability Concepts

In this section, we introduce some concepts used in our analysis of the stability of buyer alliances. The concepts we adopt lie within the framework of cooperative game theory. Before we describe the exact methodology, we will briefly try to motivate our framework.

Game-theoretical concepts of stability are usually static. In noncooperative strategic-form games, the often-used concept is the Nash equilibrium, which only considers deviations by individual players. In our setting, we assume that all buyers (coalitions) can communicate among themselves and can join or leave alliances at their will. Thus, we may expect that they will consider both unilateral and joint (multi-lateral) deviations from a given coalition structure (a partition of the set $\{1, 2, \dots, n\}$). The *strong Nash equilibrium* (SNE) in strategic form games (Aumann, 1959) admits this extension. The *coalition structure core* (Aumann and Dreze, 1974) is the cooperative analogue of the SNE. However, these solution concepts, along with the majority of solution concepts commonly used in the analysis of coalition-structures stability including the *core* (Gillies, 1959), the *bargaining set* (Aumann and Mashler, 1964) and *coalition-proof Nash equilibrium* (Bernheim et al., 1987), share the same problem that afflicts all static concepts. This can be described as follows: consider Example 1 and assume that the status-quo position is the alliance of all buyers (the grand coalition). We have shown that it is beneficial for a subset of players $\{1, 2\}$ to defect

from the grand coalition under the proportional rule. The existing static concepts will immediately conclude that the grand coalition is *not* stable. There are potentially two fundamental problems with this logic. First, does this mean that the resulting outcome, obtained by a defection of players $\{1, 2\}$, is stable? If not, why should we conclude that the move from the grand coalition will ever happen? Secondly, the static analysis does not check if a further defection will occur. It may possibly happen that an initial defection triggers a sequence of further defections that eventually leads to an outcome in which the defecting parties accrue a lower payoff than the status quo. If this were the case, farsighted players may not choose to defect in the first place, and thus an outcome which we thought was possibly not stable may actually be a candidate for stability! A static concept, by definition, does not handle such trade-offs.

A solution concept that allows players to consider multiple possible further deviations is *the largest consistent set*, introduced by Chwe (1994). It is defined below, and is used as a stability criterion in our analysis of stable alliance structures.

Any partition of N , i.e. $\mathcal{Z} = \{Z_1, \dots, Z_m\}, \cup_{i=1}^m Z_i = N, Z_j \cap Z_k = \emptyset, j \neq k$ corresponds to a *coalition structure*, \mathcal{Z} . For each buyer, let $\varphi_i^{\mathcal{Z}}$ denotes buyer i 's share of savings in the coalition structure \mathcal{Z} . Let us denote by \prec_i the players' strong preference relations, described as follows: for two coalition structures, \mathcal{Z}_1 and \mathcal{Z}_2 , $\mathcal{Z}_1 \prec_i \mathcal{Z}_2 \Leftrightarrow \varphi_i^{\mathcal{Z}_1} < \varphi_i^{\mathcal{Z}_2}$, where $\varphi_i^{\mathcal{Z}}$ is a buyer i 's allocation of saving in the coalition structure \mathcal{Z} . If $\mathcal{Z}_1 \prec_i \mathcal{Z}_2$ for all $i \in S$, we write $\mathcal{Z}_1 \prec_S \mathcal{Z}_2$. Denote by \rightarrow_S the following relation: $\mathcal{Z}_1 \rightarrow_S \mathcal{Z}_2$ if the coalition structure \mathcal{Z}_2 is obtained when S deviates from the coalition structure \mathcal{Z}_1 . We say that \mathcal{Z}_1 is *directly dominated* by \mathcal{Z}_2 , denoted by $\mathcal{Z}_1 < \mathcal{Z}_2$, if there exists an S such that $\mathcal{Z}_1 \rightarrow_S \mathcal{Z}_2$, and $\mathcal{Z}_1 \prec_S \mathcal{Z}_2$. We say that \mathcal{Z}_1 is *indirectly dominated* by \mathcal{Z}_m , denoted by $\mathcal{Z}_1 \ll \mathcal{Z}_m$, if there exist $\mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3, \dots, \mathcal{Z}_m$ and $S_1, S_2, S_3, \dots, S_{m-1}$ such that $\mathcal{Z}_i \rightarrow_{S_i} \mathcal{Z}_{i+1}$ and $\mathcal{Z}_i \prec_{S_i} \mathcal{Z}_m$ for $i = 1, 2, 3, \dots, m-1$.

A set Y is called *consistent* if $\mathcal{Z} \in Y$ if and only if for all \mathcal{V} and S , such that $\mathcal{Z} \rightarrow_S \mathcal{V}$, there is an $\mathcal{B} \in Y$, where $\mathcal{V} = \mathcal{B}$ or $\mathcal{V} \ll \mathcal{B}$, such that $\mathcal{Z} \not\prec_S \mathcal{B}$. In fact, Chwe (1994) proves the existence, uniqueness, and non-emptiness of the largest consistent set (LCS). Since every coalition considers the possibility that, once it reacts, another coalition may react, and then yet another, and so on, the LCS incorporates *farsighted* coalitional stability. The LCS describes all possible stable outcomes and has the merit of "ruling out with confidence". That is, if $\mathcal{Z} \notin$ the LCS, \mathcal{Z} cannot be stable. For a more detailed analysis of farsighted coalitional stability, see Chwe (1994). Xue (1998) has refined Chwe's LCS by introducing the notion of *perfect foresight*. Some applications of analysis of stability using Chwe's LCS criterion include Nagarajan and Sošić (2007) and Granot and Yin (2004).

4. Farsighted Stable Outcomes for the Model with Exogenous Quantities

In this section, *stability* implies stability in the farsighted sense. Thus, an outcome is stable if it belongs to the LCS. This is different from the core membership, a myopic concept.

In order to establish which outcomes can be stable, we first need to establish players' preferences for different coalition structures. We assume without loss of generality that $q_1 \leq q_2 \leq \dots \leq q_n$. We denote the consortium of all buyers by N . If, for instance, we have five buyers divided in two consortia, one containing buyers 1 and 3, and the other containing the remaining buyers, we denote it by $\{(13), (245)\}$.

4.1 Equal Allocations

We first consider equal allocation of savings, and obtain the following result.

Proposition 1 *Suppose that each buyer receives equal share of savings. Then, there is always a $k \in \{1, \dots, n\}$ such that buyer n prefers alliance $(n; n-1; \dots; n-k)$, $k \leq n-1$, to any other outcome, and each stable outcome contains alliance $(n; n-1; \dots; n-k)$.*

Note that the grand coalition of all buyers is not stable in a farsighted sense when buyers $n, n-1, \dots, n-k$ contribute significantly more than the buyers who order smaller quantities, which corresponds to the case when the equal allocations rule does not belong to the core. It is easy to derive examples in which this is true—e.g., by slightly modifying data in Example 1, say by letting $q_1 = 10, q_2 = 100$. This result comes as a natural consequence of the fact that all buyers receive an equal share of savings, hence buyers contributing a larger amount benefit by leaving the buyers who contribute less outside the purchasing consortium. We also note that one can calculate the value of k efficiently in most realistic situations. However, if the buyers' quantities are not too far apart, the grand coalition is stable, as illustrated in our next result.

Assume that the price (discount) schedule given by the seller is given by equation (1) with $\eta = -1$ and $\beta < 0$. We let $\bar{\alpha} = \alpha, \bar{\beta} = -\beta$, which leads to

$$p(q) = \bar{\alpha} - \bar{\beta}q, \quad \bar{\alpha}, \bar{\beta} > 0. \quad (3)$$

Proposition 2 *Suppose that each buyer receives equal share of savings and the quantity discount scheme is linear, $p = \bar{\alpha} - \bar{\beta}q$. Then, the grand coalition is stable if and only if*

$$q_k \geq \frac{\sum_{i>j} \sum_{j>k} q_i q_j}{(n-k) \sum_{j>k} q_j},$$

for all $1 \leq k \leq n - 2$. In particular, it is stable if

$$q_{n-2} \geq \frac{1}{4} \frac{q_{n-1} + q_n}{2}, \quad q_{n-3} \geq \frac{1}{3} \frac{q_{n-2} + q_{n-1} + q_n}{3}, \quad \text{and } q_k \geq \frac{1}{2} \frac{\sum_{j>k} q_j}{n-k} \text{ for all } 1 \leq k \leq n-4.$$

The above proposition provides first a necessary and sufficient condition, as well as a second sufficient (but not necessary) condition for the grand coalition to be stable. The former condition implies that, for instance, when $p = 500 - q$, $q_1 = 30$, $q_2 = 35$, $q_3 = 100$, and $q_4 = 200$, the grand coalition is a stable outcome. This particular example violates the latter condition. But the latter is easier to verify and interpret, which implies that, for instance, when $q_1 = 38$, $q_2 = 42$, $q_3 = 100$, and $q_4 = 200$, the grand coalition is stable. The condition also implies that, roughly speaking, if the smallest q_k is at least half of the average quantity (so that buyers' quantities are not too far apart), the grand coalition is stable.

4.2 Quantity-Based Proportional Allocation Rule

We next consider the proportional rule, which leads to the following proposition.

Proposition 3 *Suppose that each buyer receives portion of savings proportional to the quantity he contributes.*

1. *If $p'(q)q$ is increasing, then there is always a $k \in \{1, \dots, n\}$ such that buyer 1 prefers alliance $(12 \dots k)$, $k \leq n$, to any other outcome, and each stable outcome contains alliance $(12 \dots k)$;*
2. *If the quantity discount scheme is given by $p(q) = \alpha - \beta q^\eta$, $0 < \eta < 1$, and the maximum quantity ratio $\frac{q_n}{q_1} \leq \lambda(\eta)$ where $\lambda(\eta)$ is implicitly determined by*

$$\eta(1 + \lambda)((1 + \lambda)^\eta - \lambda^\eta) - \lambda^\eta + 1 = 0, \tag{4}$$

then there is always a $k \in \{1, \dots, n\}$ such that buyer n prefers alliance $(n; n-1; \dots; n-k)$, $k \leq n-1$, to any other outcome, and each stable outcome contains alliance $(n; n-1; \dots; n-k)$;

3. *If the quantity discount scheme is linear, $p(q) = \bar{\alpha} - \bar{\beta}q$, then the grand coalition is the only stable outcome.*

The assumption that $p'(q)q$ is increasing holds for many forms of a decreasing convex function $p(q)$, such as $a + b/\ln(q)$, or the function described by (1) when $\eta > 0$ (the opposite of case 2 above). Note that the grand coalition is not stable in a farsighted sense when $p'(q)q$ is increasing and buyers $1, 2, \dots, k$ contribute significantly less than the buyers who order larger quantities,

which corresponds to the case when the proportional rule does not belong to the core. It is easy to derive examples in which this is true—e.g., for the data in Example 1. This result comes as a natural consequence of the fact that buyers receive allocations proportional to their contributions, hence buyers with small contributions benefit by leaving the buyers who participate more outside the purchasing alliance.

In cases 2 and 3 above we obtain an opposite result, in which buyers prefer alliances with larger buyers and as a result the grand coalition can be stable. The threshold function $\lambda(\eta)$, determined by expression (4), is illustrated in Figure 4.2. We observe that $\lambda(\eta)$ is increasing in η , with $\lim_{\eta \rightarrow 0} \lambda(\eta) = 1$ and $\lim_{\eta \rightarrow 1} \lambda(\eta) = +\infty$.

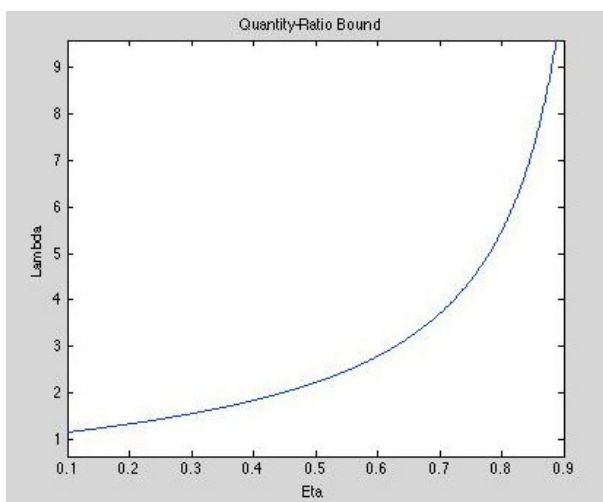


Figure 1: Maximum quantity ratio for sustaining the grand coalition.

4.3 Shapley Value Allocations

We next consider the Shapley value, and have the following result.

Proposition 4 *When buyers receive Shapley value allocations, the grand coalition is the only stable outcome.*

Proof: Rosenthal (1990) shows that the Shapley value on convex games satisfies population-monotonicity. In other words, addition of new players expands opportunities of all players, and all players in the new game are better off. Thus, each player prefers the grand coalition to any other outcome, and it is easy to show that the grand coalition is stable in the farsighted sense. ■

4.4 Comparisons and Conclusions

As shown above, when buyers’ contributions are close to each other, the grand coalition is stable regardless of which allocation rule is used. The exact meaning of “closeness” is to be understood from the statements of the preceding results. The results indicate that when players are significantly heterogeneous, the equal allocations and proportional rule, which are popular in practice, will not lead to stable GPOs. On the other hand, when buyers’ contributions are significantly different, overall savings will be maximized under the Shapley allocation rule, because the buyers in that case will not have an incentive to defect and form subcoalitions. Thus our central message—using the Shapley value to allocate savings to the members of a GPO is essentially robust to stability concepts, heterogeneity of buyer requirements, and the perception of fairness among players.

5. Price-Dependent Quantities with Linear Discount Scheme

In the previous sections, we assumed that each buyer determines the quantity he wants to purchase before alliance formation, and the discount schedule does not have an impact on the actual quantity purchased. We referred to this as the *exogenous* model, in which purchasing requirements were a given input. In the remainder of the paper, we analyze situations where each buyer determines his purchasing requirements based both on a demand curve that his firm faces and on its supply curve (which is nothing but the effective discount schedule based on his membership in an alliance). To the best of our knowledge, this is the first paper considering such scenarios.

In this section, we begin with perhaps the simplest instance of this situation. We assume that all buyers charge the same retail price², r . This applies in several settings in which, due to competition, there is a mature market for the product in question and firms see little differences in retail price. Our objective here is to model the purchasing requirement of a buyer as a function of the discount schedule he sees from the seller. Despite the fact that final retail prices are fixed due to market factors, one can intuit the buyer’s requirements as a function of his purchase price. Reasonable explanations for this include artifacts of the final demand that we do not model, such as sales efforts in a store, which are usually proportional to the margin on a product, demand uncertainty, wherein service levels are affected by margins, etc. That is, retailers routinely push commodity products whose margins are high, thus influencing the final demand. In a later section, we relax this assumption, and analyze a full-blown model of downstream buyer demand. For tractability, we assume the often-used linear demand model for the firm that buyer i manages. Thus, we assume

²We note that our result is robust to small differences in retail price among sellers.

the quantity purchased by buyer i (alternately, his purchasing requirement) depends on the seller's price in a linear fashion,

$$D_i(p) = q_i = a_i - b_i p.$$

We further assume that the price (discount) schedule is given by (3),

$$p(q) = \bar{\alpha} - \bar{\beta} q, \quad \bar{\alpha}, \bar{\beta} > 0.$$

It is easy to evaluate the purchase quantity of each buyer³. An individual buyer, i , who places orders individually and is not a member of any consortia orders

$$\hat{q}_i = \frac{a_i - b_i \bar{\alpha}}{1 - \bar{\beta} b_i} \quad (5)$$

and pays

$$\hat{p}_i = p(\hat{q}_i) = \frac{\bar{\alpha} - \bar{\beta} a_i}{1 - \bar{\beta} b_i}. \quad (6)$$

If the buyers form coalitions and place their orders jointly, we denote the quantity ordered by coalition S by \hat{Q}^S and the corresponding price by \hat{p}^S . It is easy to verify that

$$\hat{Q}^S = \frac{\sum_{i \in S} (a_i - b_i \bar{\alpha})}{1 - \bar{\beta} \sum_{i \in S} b_i}, \quad \hat{p}^S = \frac{\bar{\alpha} - \bar{\beta} \sum_{i \in S} a_i}{1 - \bar{\beta} \sum_{i \in S} b_i}.$$

We will denote the quantity allocated to each buyer under price \hat{p}^N (that is, when all buyers form the grand coalition) by $\hat{Q}_i^N = a_i - b_i \hat{p}^N$.

Because we need $\hat{q}_i > 0$ and $\hat{p}_i > 0$, equations (5) and (6) imply that we need one of the following two cases to hold:

1. $1 - \bar{\beta} b_i > 0, a_i - b_i \bar{\alpha} > 0, \bar{\alpha} - \bar{\beta} a_i > 0$; or
2. $1 - \bar{\beta} b_i < 0, a_i - b_i \bar{\alpha} < 0, \bar{\alpha} - \bar{\beta} a_i < 0$.

First, suppose that the second case occurs (that is, $\bar{\beta} b_i > 1$) and consider the following example.

EXAMPLE 2. Suppose that the players are symmetric ($a_i = a_j = a, b_i = b_j = b \forall i, j \in N$), and that $a_i - b_i \bar{\alpha} < 0 \forall i, \bar{\alpha} - \bar{\beta} \sum_{i \in N} a_i < 0$, and $\beta \sum_{i \in N} b_i > 1$. Then,

$$\hat{Q}^N = \frac{\sum_{i \in S} (a_i - b_i \bar{\alpha})}{1 - \bar{\beta} \sum_{i \in S} b_i} = \frac{n(a - b \bar{\alpha})}{1 - \bar{\beta} n b} = n \hat{q} \frac{\bar{\beta} b - 1}{\bar{\beta} n b - 1} < n \hat{q},$$

³We note, for instance, that if a buyer orders $q = \bar{\alpha}/\bar{\beta}$, he would pay zero for every unit ordered, and he could maximize his profit by selling only a part of his inventory. As will be seen later, we impose a condition that the seller always charges a positive price, hence each buyer sells his entire inventory in the optimal solution. We discuss later the case in which this assumption is relaxed.

hence the quantity purchased decreases as a result of cooperative buying. \square

From our discussion, we notice that the linear discounting scheme described by (3) does not perform well when the range of prices is large, which is likely to happen when $\beta \sum_{i \in N} b_i > 1$. As seen from the Example 2 above, such situations may lead to inconsistent results in which cooperating retailers order less. This, though analytically possible, is unrealistic in practice. Thus, hereafter we consider the conditions from the first case and make the following assumptions.

Assumption 1 *When the seller offers a linear discount scheme, $p(q) = \bar{\alpha} - \bar{\beta}q$, we assume that there is a positive demand at a highest price, $D_i(\bar{\alpha}) = a_i - b_i\bar{\alpha} > 0$, and that the supplier charges a positive price for the highest demand, $p(\sum_{i \in N} a_i) = \bar{\alpha} - \bar{\beta} \sum_{i \in N} a_i > 0$.*

We note that Assumption 1 implies $\bar{\beta} \sum_{i \in N} b_i < 1$:

$$1 - \bar{\beta} \sum_{i \in N} b_i = \frac{\sum_{i \in N} a_i - \bar{\alpha} \sum_{i \in N} b_i}{\sum_{i \in N} a_i} + \sum_{i \in N} b_i \cdot \frac{\bar{\alpha} - \bar{\beta} \sum_{i \in N} a_i}{\sum_{i \in N} a_i} > 0.$$

This leads to the following result.

Proposition 5 *Under Assumption 1, the price decreases as a result of cooperative buying, $\hat{p}^S < \min\{\hat{p}_i, i \in S\}$, and the total quantity purchased increases, $\hat{Q}^S > \sum_{i \in S} \hat{q}_i$.*

Proof of Proposition 5: We first consider the total quantities purchased by the members of coalition S with and without cooperation:

$$\begin{aligned} \hat{Q}^S - \sum_{i \in S} \hat{q}_i &= \frac{\sum_{i \in S} (a_i - b_i \bar{\alpha})}{1 - \bar{\beta} \sum_{j \in S} b_j} - \sum_{i \in S} \frac{a_i - b_i \bar{\alpha}}{1 - \bar{\beta} b_i} \\ &= \sum_{i \in S} (a_i - b_i \bar{\alpha}) \left(\frac{1}{1 - \bar{\beta} \sum_{j \in S} b_j} - \frac{1}{1 - \bar{\beta} b_i} \right) > 0. \end{aligned}$$

Thus, the quantity purchased increases as a result of cooperative buying. Consequently, the price seen by the coalition members is lower than the price paid by any of the members acting independently. \blacksquare

The implication from Proposition 5 is that the quantity purchased by each buyer increases as a result of cooperation. This further implies that the total cost incurred by a buyer may actually increase after cooperation (simply because he orders more units), as illustrated in the following example.

i	$D_i(p)$	\hat{q}_i	\hat{p}_i	$\hat{q}_i\hat{p}_i$	\hat{Q}_i	\hat{p}^N	$\hat{Q}_i\hat{p}^N$
1	$200 - 4p$	50.0	37.5	1,875	73	32	2,314
2	$250 - 5p$	66.7	36.7	2,444	91	32	2,893

Table 2: Ordering quantities, unit prices, and total costs before and after cooperation

EXAMPLE 3. Suppose that $p(q) = 40 - 0.05q$, and that $N = \{1, 2\}$. Table 2 provides ordering quantities, unit prices, and total costs before and after cooperation for two buyers with demand functions $D_i(p)$. \square

From a practical perspective, this means that we need to take additional care in apportioning the overall benefits of joining a consortia to individual members. This leads us to introduce a new measure for benefits from cooperation, i.e., the total profits obtained by the buyers. As we assume that all buyers charge the same retail price, r , a buyer, i , generates profit $(r - \hat{p}_i)\hat{q}_i$. To insure that each buyer makes a positive margin, $r \geq \hat{p}_i$, (3) implies that we should use the following assumption.

Assumption 2 *When the seller offers a linear discount scheme, $p(q) = \bar{\alpha} - \bar{\beta}q$, we assume that $r > \bar{\alpha}$, which insures that each buyer makes a positive margin.*

Thus, instead of the savings game, we now consider the profit game, and define the values on the coalitions as follows. If all buyers cooperate, the total increase in profit is given by

$$\begin{aligned} \hat{v}(N) &= (r - \hat{p}^N)\hat{Q}^N - \sum_{j=1}^n (r - \hat{p}_j)\hat{q}_j \\ &= \left(r - \frac{\bar{\alpha} - \bar{\beta} \sum_{i \in N} a_i}{1 - \bar{\beta} \sum_{i \in N} b_i} \right) \frac{\sum_{i \in N} (a_i - b_i \bar{\alpha})}{1 - \bar{\beta} \sum_{i \in N} b_i} - \sum_{j \in N} \left(r - \frac{\bar{\alpha} - \bar{\beta} a_j}{1 - \bar{\beta} b_j} \right) \frac{a_j - b_j \bar{\alpha}}{1 - \bar{\beta} b_j}. \end{aligned}$$

The profit of a coalition S which yields its value function is derived similarly and is given by

$$\hat{v}(S) = (r - \hat{p}^S)\hat{Q}^S - \sum_{j \in S} (r - \hat{p}_j)\hat{q}_j. \quad (7)$$

We can now state our first result for the setting with endogenous order quantities. The proof is relegated to the appendix.

Theorem 1 *Under Assumption 1, the profit game is convex.*

If a buyer, i , belongs to the grand coalition, N , the increases in profit he observes due to alliance membership can be written as $(r - \hat{p}^N)\hat{Q}_i^N - (r - \hat{p}_i)\hat{q}_i$. His contribution to the grand coalition may be written as $(r - \hat{p}^N)\hat{Q}^N - (r - \hat{p}_i)\hat{q}_i - (r - \hat{p}^{N \setminus i})\hat{Q}^{N \setminus i}$, or alternatively,

$$\left[(r - \hat{p}^N)\hat{Q}_i^N - (r - \hat{p}_i)\hat{q}_i \right] + \sum_{j \neq i} \left[(r - \hat{p}^N)\hat{Q}_j^N - (r - \hat{p}^{N \setminus i})\hat{Q}_j^{N \setminus i} \right].$$

Thus, besides observing an increase in profits of $(r - \hat{p}^N)\hat{Q}_i^N - (r - \hat{p}_i)\hat{q}_i$ himself, which is generated by both himself and the remaining buyers, i also contributes an increase in profits of $\sum_{j \neq i} \left[(r - \hat{p}^N)\hat{Q}_j^N - (r - \hat{p}^{N \setminus i})\hat{Q}_j^{N \setminus i} \right]$ for the remaining alliance members. Similarly as before, it is easy to generate examples wherein a company that contributes more receives lower profit increase than the companies that contribute less to the value of \hat{Q} if each buyer pays the same price.

We assume, as before, without loss of generality that $\hat{q}_1 \leq \hat{q}_2 \leq \dots \leq \hat{q}_n$, and consider the three allocation rules analyzed earlier.

5.1 Equal Allocations

Under equal allocation of benefits,

$$\hat{\varphi}_i^E = \frac{(r - \hat{p}^N)\hat{Q}^N - \sum_{j=1}^n (r - \hat{p}_j)\hat{q}_j}{n},$$

and buyer i receives profit

$$\begin{aligned} \pi^E(\mathbf{q}) &= (r - \hat{p}_i)\hat{q}_i + \frac{(r - \hat{p}^N)\hat{Q}^N - \sum_{j=1}^n (r - \hat{p}_j)\hat{q}_j}{n} \\ &= \frac{\sum_{j=1}^n \left[(r - \hat{p}_i)\hat{q}_i + (r - \hat{p}^N)\hat{Q}_i^N - (r - \hat{p}^N)\hat{Q}_i^N - (r - \hat{p}_j)\hat{q}_j + (r - \hat{p}^N)\hat{Q}_j^N \right]}{n} \\ &= \frac{\sum_{j \neq i} \left\{ [(r - \hat{p}_i)\hat{q}_i - (r - \hat{p}^N)\hat{Q}_i^N] - [(r - \hat{p}_j)\hat{q}_j - (r - \hat{p}^N)\hat{Q}_j^N] \right\}}{n} + (r - \hat{p}^N)\hat{Q}_i^N. \end{aligned}$$

Proposition 5 implies that we can apply an analysis similar to that used in the previous section to the equal allocation of savings, which leads to the following result.

Proposition 6 *Suppose that each buyer receives equal share of benefits and Assumption 1 holds. Then, there is always a $k \in \{1, \dots, n\}$ such that buyer n prefers alliance $(n; n-1; \dots; n-k)$, $k \leq n-1$, to any other outcome, and each stable outcome contains alliance $(n; n-1; \dots; n-k)$.*

5.2 Quantity-Based Proportional Allocation Rule

Under quantity-based proportional allocation rule,

$$\hat{\varphi}_i^P = \frac{\hat{Q}_i^N \left[(r - \hat{p}^N) \hat{Q}^N - \sum_{j=1}^n (r - \hat{p}_j) \hat{q}_j \right]}{\hat{Q}^N},$$

and buyer i receives profit

$$\begin{aligned} \pi^P(\mathbf{q}) &= (r - \hat{p}_i) \hat{q}_i + \frac{\hat{Q}_i^N \left[(r - \hat{p}^N) \hat{Q}^N - \sum_{j=1}^n (r - \hat{p}_j) \hat{q}_j \right]}{\hat{Q}^N} \\ &= (r - \hat{p}^N) \hat{Q}_i^N + \sum_{j=1}^n \frac{\hat{Q}_j^N (r - \hat{p}_i) \hat{q}_i - \hat{Q}_i^N (r - \hat{p}_j) \hat{q}_j}{\hat{Q}^N} \end{aligned}$$

Unlike the case with exogenous demand, a buyer who contributes the most can receive more or less than $(r - \hat{p}^N) \hat{Q}_i^N$, as seen in the following example.

EXAMPLE 4. Suppose that $r = 50$, $p(q) = 40 - 0.05q$, and that $N = \{1, 2, 3\}$. Table 3 gives ordering quantities and unit prices before and after cooperation for three buyers with demand functions $D_i(p)$.

i	$D_i(p)$	\hat{q}_i	\hat{p}_i	\hat{Q}_i	\hat{p}^N
1	$200 - 4p$	50.0	37.5	94	26.6
2	$60 - 0.1p$	56.3	37.2	57	26.6
3	$250 - 5p$	66.7	36.7	117	26.6

Table 3: Ordering quantities and unit prices before and after cooperation

The profit observed by buyer 3 when he acts alone is 889, it is 2,737 when each buyer pays the same price, and it decreases to 2,650 when the buyers receive proportional allocations. \square

Due to complexity of expressions, we are unable to obtain general closed-form results for stable alliances under the proportional rule. However, we are able to derive several results for some special classes of buyers.

Suppose that the buyers' possess the following property:

$$a_i \geq a_j \implies \frac{a_i}{b_i} \geq \frac{a_j}{b_j}. \quad (8)$$

It is easy to see that whenever condition (8) holds, the graphs describing buyers' purchasing quantities, $a_i - b_i p$, do not intersect for $p \geq 0$. An immediate consequence of this property is the

preservation of the order between buyers' ordering quantities—if buyer i orders more than buyer j before an GPO is formed, he orders more than buyer j after an GPO is formed (note that this is not true if (8) does not hold; for instance, in example 4 buyers 1 and 2 order 50 and 56.3, resp., before a GPO is formed, while after a formation of a two-member GPO they order 94 and 57, resp.). However, as illustrated in the following example, the allocation to buyer i may decrease if the quantity ordered by buyer j increases under this scenario. This assumption is not overly restrictive and simply falls out of the degree of heterogeneity that characterizes individual consumer choices from which the demand model is constructed.

EXAMPLE 5. Suppose that $r = 100$, $p(q) = 40 - 0.05q$, and that $N = \{1, 2\}$. Table 4 gives ordering quantities and unit prices before and after cooperation for two buyers with demand functions $D_i(p)$.

i	$D_i(p)$	\hat{q}_i	\hat{p}_i	\hat{Q}_i	\hat{p}^N	φ_i^P
1	$200 - 4p$	50	37.50	69.6	26.1	1,738
2	$300 - 4.5p$	155	32.26	208.7	26.1	5,214

Table 4: Ordering quantities and unit prices before and after cooperation

Now, suppose that buyer 1 instead faces function $D'_1(p) = 210 - 3.5p$, while $D'_2(p) = D_2(p)$. Table 5 gives ordering quantities and unit prices before and after cooperation for two buyers with demand functions $D'_i(p)$. The quantity ordered by buyer 1 increases, $q'_1 > q_1$, while demand

i	$D'_i(p)$	\hat{q}'_i	\hat{p}'_i	\hat{Q}'_i	\hat{p}'^N	φ'^P_i
1	$210 - 3.5p$	85	35.76	116.7	24.17	2,975
2	$300 - 4.5p$	155	32.26	200.0	24.17	5,099

Table 5: Ordering quantities and unit prices before and after cooperation

functions $D_1(p), D_2(p)$ and $D'_1(p)$ do not intersect for $p \geq 0$. Note that the quantity ordered by buyer 2 in GPO decreases, $Q'_2 < Q_2$, and buyer 2 receives a smaller share of the additional profit after joining the coalition with a buyer who orders more, $\varphi'^P_2 < \varphi^P_2$. \square

As illustrated in the example above, even if the other buyer increases his ordering quantity, the allocation to buyer i may decrease if the slope of the quantity ordering function, b_j , decreases. However, if an increase in a_j implies an increase in b_j , we have the following result.

Proposition 7 *Suppose that each buyer receives share of benefits proportional to the quantity he contributes, that $a_i \geq a_j \implies b_i \geq b_j$ for all i, j , and Assumption 1 holds. Then, there is always a $k \in \{1, \dots, n\}$ such that buyer n prefers alliance $(n; n - 1; \dots; n - k)$, $k \leq n - 1$, to any other outcome, and each stable outcome contains alliance $(n; n - 1; \dots; n - k)$.*

As shown in the proposition, when the buyers face linear discount scheme, the model in which the amount ordered is influenced by the price paid preserves the flavor of the model in which the quantity ordered by buyers is not influenced by the actual price (e.g., components/items are utilized at a steady rate)—that is, buyers prefer to join partners whose ordering quantities are large.

5.3 Shapley Value Allocations

Because of Theorem 1, an analysis similar to that in Section 4 implies the following result for the Shapley value.

Proposition 8 *When buyers receive Shapley value allocations and Assumption 1 holds, the grand coalition is the only stable outcome.*

5.4 Comparisons and Conclusions

The analysis above indicates that our main results from the model with exogenous demand extends to the case with price-dependent demand under linear quantity-discount scheme when Assumption 1 holds and the buyers use equal allocations or Shapley value allocations, while we can see a significant change in stable outcomes under the proportional rule. That is, for the linear demand model, using the Shapley value allocation stands the best chance of sustaining the GPO both due to its stability and the apparent fairness of the calculation. However, if one of the simpler allocation rules is used (equal allocations or proportional allocations), then we can observe stable alliances which contain several largest buyers. In fact, we can easily generate examples such that under equal allocations and the quantity based proportional rules one can confidently rule out the grand coalition as being stable (i.e., $k < n - 1$).

We conclude our analysis by noting that allowing for arbitrary discounting schemes leads to complex expressions and non-convex games, as illustrated in the following section.

6. Extensions of the Model with Price-Dependent Quantities

In this section, we discuss two possible extensions of our model analyzed in the previous section—first, the case in which the discount scheme from the seller is more general, and second, the case in which each buyer determines both his optimal order quantity as well as sets the optimal selling price in his firm.

6.1 Arbitrary Discount Scheme

In this section, we do a simple analysis using examples to see what happens when quantities are endogenously obtained by buyers when facing a more general quantity discount schedule than the one analyzed in the previous section. The purpose of this section is to simply illustrate that when discount schedules get complicated, conditions are not conducive for stable GPOs to exist. Specifically, when the discount scheme is not linear, the resulting profit game does not have nice properties as before (e.g., Theorem 1).

To illustrate this point, assume that the price is given by equation (1) with $\eta = 0.5$, $p(q) = \alpha + \beta q^{-0.5}$, $\alpha, \beta > 0$. To obtain a solution, we need to solve equation

$$\frac{a_i - q_i}{b_i} = \alpha + \beta q_i^{-0.5},$$

which corresponds to

$$\varsigma(q_i) = a_i - \alpha b_i - q_i - \frac{\beta b_i}{\sqrt{q_i}} = 0. \quad (9)$$

Let us denote $x_i = \sqrt{q_i}$. Then, (9) can be rewritten as

$$-x_i^3 + (a_i - \alpha b_i)x_i - \beta b_i = 0,$$

which is a polynomial that can have either one or three zeros. The local extremes are obtained when

$$-3x_i^2 + a_i - \alpha b_i = 0,$$

and we get local maximum for $x_i > 0$. Thus, the local maximum is obtained for $\bar{q}_i = (a_i - \alpha b_i)/3$.

If we evaluate $\varsigma(\bar{q}_i)$, we obtain

$$\varsigma(\bar{q}_i) = a_i - \alpha b_i - \frac{a_i - \alpha b_i}{3} - \frac{\beta b_i}{\sqrt{\frac{a_i - \alpha b_i}{3}}}.$$

If $\varsigma(\bar{q}_i) < 0$, then $\varsigma(q) < 0$ for all $q > \bar{q}_i$, and (9) does not have a solution for $q > \bar{q}_i$. Because $\varsigma(x_i)$ achieves minimum for $x_i < 0$, we can find a solution only if $\varsigma(\bar{q}_i) > 0$, which corresponds to

$$\beta b_i < \left(\frac{a_i - \alpha b_i}{3} \right)^{\frac{3}{2}}. \quad (10)$$

Thus, hereafter we assume that (10) holds. In addition, equation (9) can have two different solutions. Note that the seller's profit increases with q , so he picks the lower of the two corresponding prices. Let denote this price by \hat{p}_i , and the corresponding quantity $D_i(\hat{p}_i) = \hat{q}_i$.

Unfortunately, when the discount scheme is not linear, the resulting profit game does not have nice properties as before, as can be seen from the following example.

EXAMPLE 6. Suppose that $p(q) = 10 + 30q^{-0.5}$, and that $N = \{1, 2, 3\}$. In addition, suppose $D_1(p) = 51 - 2p$, $D_2(p) = 32 - p$, and $D_3(p) = 100 - 2p$, and that the unit retail price equals $r = 40$. Then, Table 2 gives us ordering quantities, prices, and corresponding profits for all different coalitions.

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1, 2\}$	$\{1, 3\}$	$\{2, 3\}$	$\{1, 2, 3\}$
\hat{p}^S	17.5	18.0	13.7	14.8	13.0	13.1	12.3
\hat{q}^S	16	14	73	39	99	93	119
$\hat{v}(S)$	0	0	0	302	386	260	660

Table 6: Values of different coalitions

First, let $i = 3, S = \{1\}, T = \{1, 2\}$. Then, $S \subset T \subseteq N \setminus \{i\}$ and

$$\hat{v}(\{1, 2, 3\}) - \hat{v}(\{1, 2\}) = 660 - 302 = 358; \quad \hat{v}(\{1, 3\}) - \hat{v}(\{1\}) = 386 - 0 = 386,$$

hence $\hat{v}(T \cup \{i\}) - \hat{v}(T) < \hat{v}(S \cup \{i\}) - \hat{v}(S)$. On the other hand, if we let $S = \{2\}$, we have $\hat{v}(\{2, 3\}) - \hat{v}(\{2\}) = 260 - 0 = 260$, hence $\hat{v}(T \cup \{i\}) - \hat{v}(T) > \hat{v}(S \cup \{i\}) - \hat{v}(S)$. \square

This analysis can be summarized in the following observation.

Observation 1 *The profit game with endogenous quantities is, in general, neither convex nor concave.*

The message from this calculation is that, when one looks at general endogenous setting, the games under question lose several of the nice properties that facilitate an elegant analysis. Games such as these may neither be convex nor concave. In fact, the above example also demonstrates

that under most reasonable allocation rules such as the 3 rules discussed in this work, the grand coalition of all players will not be farsighted stable. The implication is that in the presence of non-linear schedules and demands (for instance, think of a demand with constant elasticity), one needs to be very careful in prescribing allocation rules. In general, allocation rules required to preserve the alliance may be far too complicated to implement, and in some instances may not even exist. Thus, perhaps an important insight is that, in such instances, it may well be in the health of the GPO to negotiate less complicated discount schedules that may lead to slightly lower overall savings, but a stable GPO with a greater chance of success.

6.2 Linear Discount Scheme with Optimal Retail Prices

In this section, we propose a more complex model to endogenously determine the purchasing requirements of each buyer when facing a discount schedule. Each buyer i is facing the same wholesale-price quantity discount schedule as before, described by (3), $p(q) = \bar{\alpha} - \bar{\beta}q$. However, we now drop the assumption that his selling price (retail price) is predetermined. Rather, we now look at price-setting firms determining their purchasing requirements based on the discount schedule and their downstream demand, which is directly affected by the price they set in the market. This assumption models situations in which the firms are genuine price-setters and margins may not be the only driving force behind their decisions. Informally speaking, one may argue that the products under question are less influenced by market competition and are thus less mature. We now model the actual downstream demand of buyer i as $D_i(r) = a_i - b_i r$, linking it to his retail price r .

Our analysis is as follows. In stage 1, the buyer orders q_i^p from the wholesaler. In stage 2, he sells $q_i^r (\leq q_i^p)$ to the consumers. Without collaboration, buyer i has full control over q_i^p and q_i^r . Note that these two stages were absent in our previous analysis. This is because, when one allows for a firm demand as above, one has to allow for the possibility that a buyer may purchase a particular quantity but set a price that does not clear all the inventory he purchased. Formally, the seller's revenue function, $q(\bar{\alpha} - \bar{\beta}q)$, is maximized at $\bar{\alpha}/(2\bar{\beta})$; thus, it is natural to assume that the wholesaler will not sell more than $\bar{\alpha}/(2\bar{\beta})$ (or she will use a different price schedule after quantity

$\bar{\alpha}/(2\bar{\beta})$). Buyer i 's problem can then be summarized as:

$$\max_{q_i^r, q_i^p} \Pi(q_i^r, q_i^p) = r q_i^r - w q_i^p \quad (11a)$$

$$\text{s.t. } r = \frac{a_i - q_i^r}{b_i}, \quad (11b)$$

$$p = \bar{\alpha} - \bar{\beta} q_i^p, \quad (11c)$$

$$q_i^r \leq q_i^p \leq \bar{\alpha}/(2\bar{\beta}). \quad (11d)$$

By substitution, the objective function (11a) becomes:

$$\Pi(q_i^r, q_i^p) = \frac{a_i - q_i^r}{b_i} q_i^r - (\bar{\alpha} - \bar{\beta} q_i^p) q_i^p. \quad (12)$$

Given any q_i^r , the function (12) is convex in q_i^p , so the optimal q_i^p must be a boundary point, i.e., either q_i^r or $\bar{\alpha}/(2\bar{\beta})$. Because $-(\bar{\alpha} - \bar{\beta} q_i^p) q_i^p$ is decreasing in q_i^p over $[0, \bar{\alpha}/(2\bar{\beta})]$, we must have $q_i^{p*} = q_i^r$ (we will ensure $q_i^r \leq \bar{\alpha}/(2\bar{\beta})$ later). Thus, by substitution, the objective function becomes:

$$\Pi(q_i^r) = \frac{a_i - q_i^r}{b_i} q_i^r - (\bar{\alpha} - \bar{\beta} q_i^r) q_i^r = \left[\left(\frac{a_i}{b_i} - \bar{\alpha} \right) - \left(\frac{1}{b_i} - \bar{\beta} \right) q_i^r \right] q_i^r.$$

We continue to assume that Assumption 1 holds; thus, $\Pi(q_i^r)$ is concave in q_i^r and is therefore optimized by

$$q_i^* = \frac{1}{2} \frac{\frac{a_i}{b_i} - \bar{\alpha}}{\frac{1}{b_i} - \bar{\beta}} = \frac{1}{2} \frac{a_i - \bar{\alpha} b_i}{1 - \bar{\beta} b_i}. \quad (13)$$

The second inequality of Assumption 1 ensures that $q_i^* \leq \bar{\alpha}/(2\bar{\beta})$.

Interestingly, the market still clears at the optimal solution ($q_i^{p*} = q_i^{r*} = q_i^*$), and the optimal order quantity for buyer i given by (13) is exactly half of that obtained when the retail price is predetermined, (5). The seller's price p_i^* and retail price r_i^* can be easily computed from q_i^* :

$$p_i^* = \bar{\alpha} - \bar{\beta} q_i^* = \bar{\alpha} - \frac{\bar{\beta}}{2} \frac{a_i - \bar{\alpha} b_i}{1 - \bar{\beta} b_i}, \quad r_i^* = \frac{a_i - q_i^*}{b_i} = \frac{a_i}{b_i} - \frac{1}{2 b_i} \frac{a_i - \bar{\alpha} b_i}{1 - \bar{\beta} b_i}.$$

In addition, the difference between the retail price and the seller's price is given by

$$r_i^* - p_i^* = \frac{a_i}{b_i} - \bar{\alpha} + \frac{1}{2} \left(\bar{\beta} - \frac{1}{b_i} \right) \frac{a_i - \bar{\alpha} b_i}{1 - \bar{\beta} b_i} = \frac{1}{2} \left(\frac{a_i}{b_i} - \bar{\alpha} \right) > 0.$$

Suppose now that the buyers form coalitions. For the ensuing discussion, we assume that collaborations are "tight", i.e., that the buyers not only jointly purchase from the seller, but also agree to set the same price. The most common examples of such ventures are cooperatives in various industries that jointly procure raw materials and set prices collusively. For several such examples see Nagarajan and Sošić (2007).

The optimal solution for such a collaboration can be easily obtained from that of an individual buyer above—simply define $a_S = \sum_{i \in S} a_i$ and $b_S = \sum_{i \in S} b_i$, and replace the index i by S throughout the previous section. It is easy to verify that Proposition 5 still holds. However, under this model a GPO can actually decrease the total profit generated by its participating members, as illustrated in the following example.

EXAMPLE 7. Suppose that $p(q) = 40 - 0.05q$, and that $N = \{1, 2\}$. Table 7 gives ordering quantities, unit prices, and profits before cooperation for two buyers with demand functions $D_i(r)$.

i	$D_i(r)$	q_i^*	p_i^*	r_i^*	π_i^*
1	$200 - 4r$	25.00	38.75	43.75	125
2	$100 - r$	31.58	38.42	68.42	947

Table 7: Ordering quantities, unit prices, and profits before cooperation

If the two buyers form a GPO, then $Q^* = 66.67, p^* = 36.67, r^* = 46.67$, and the new total profit is 666.67, which is even less than the lone buyer 2 was making on its own. \square

Thus, when the buyers face arbitrary linear demand functions, tight collaboration may make them worse off, although the quantity ordered increases and the price charged by the seller decreases. In addition, the game in this case loses the convexity property. The above example, in fact shows that the grand coalition is not stable under such a situation. However, we can find instances under which our earlier results carry over.

Proposition 9 *Suppose that buyers optimally determine their retail price, that $\frac{a_i}{b_i} = C$ for all i , and that the buyers' collaboration is tight.*

1. *Suppose that either each buyer receives (i) an equal share of benefits, or (ii) a share of benefits proportional to the quantity he contributes. Then, there is always a $k \in \{1, \dots, n\}$ such that buyer n prefers alliance $(n; n-1; \dots; n-k), k \leq n-1$, to any other outcome, and each stable outcome contains alliance $(n; n-1; \dots; n-k)$.*
2. *When buyers receive Shapley value allocations, the grand coalition is the only stable outcome.*

The condition in Proposition 9 simply indicates that although players are heterogeneous, the price at which they see their demand vanish is identical. This assumption is no stronger than the ones that are often used in the marketing and mechanism design literature, where the valuation of

the highest consumer is the same across various players. With this assumption, our findings from the previous sections seem to be robust. The Shapley value seems to be the best allocation rule that one needs to prescribe. Indeed, we note that this condition may seem somewhat restrictive. Our numerical analysis indicates that the results of Proposition 9 hold under some more general cases (similar to those in Proposition 7), but we were not able to show this analytically.

The game for a loose collaboration, in which each buyer determines his own selling price, becomes even more complicated, hence none of the nice structural properties are carried over. While our numerical analysis indicates that results like Proposition 9 continue to hold, the expressions become too complex for analytical evaluation. Thus, in the settings in which the products are less mature or when the retailers face significant pricing decisions, it appears that the stability of purchasing alliances is much harder to achieve. The main lesson from this analysis is that as the number of levers (such as their retail price) become an important decision for buyers, the complexity of managing a GPO dramatically increases. Another way of understanding this is that when buyers face significant market factors that may lead them to price-setting behavior, their strategic decisions may not be aligned with the share of cost savings from being in a GPO, and in order to sustain a GPO, simple rules similar to the ones analyzed earlier may not be sufficient.

7. Conclusions and Future Research

There are a few avenues for extending and testing some of the conclusions in our work. Our analysis seems to generate the hypothesis that, when facing heterogenous contributions in a purchasing consortia, allocating the gains on the basis of the marginal value of a member's contribution is a good idea. This hypothesis, with some work, can be tested in a laboratory or through field studies. More importantly, from a computational perspective, the allocations for the savings game can be computed relatively easily. Thus, the recommendations in our work can be easily implemented. From an analytical perspective, modeling endogenous decisions seems to become intractable very quickly. One approach to perform stability analysis in this regard may be to take the view of requiring robustness with respect to perturbations (a fuzzy stability concept) in the final market price. This analysis is not easy and calls for techniques that are not well developed yet. We leave such musings to future research in this area.

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Appendix

Proof of Proposition 1: Because each buyer receives an equal share of savings, buyers prefer alliances with players who contribute more to the total quantity purchased. If we consider two-buyer alliances, it is easy to observe that each buyer $i \neq n$ prefers an alliance with buyer n , because he receives the largest allocation of savings. Buyer n prefers alliance with $n - 1$, which maximizes his share of savings. In three-buyer alliances, each buyer $i \neq n, n - 1$ prefers an alliance with buyers n and $n - 1$, while buyers n and $n - 1$ prefer alliance $(n; n - 1; n - 2)$. A similar analysis holds for any alliance size. Thus, buyer n prefers one of the following alliances: $(n; n - 1), (n; n - 1; n - 2), \dots, (n; n - 1; \dots; 2), N$.

Suppose that buyer n prefers the alliance $(n; n - 1)$ to any other outcome. Because buyers n and $n - 1$ receive equal shares of savings and have equal preferences for alliances of all sizes, this implies that buyer $n - 1$ also prefers the alliance $(n; n - 1)$ to any other outcome. As their savings to not depend on actions of other buyers, the other buyers may not induce buyers n and $n - 1$ to defect from this alliance, hence every stable outcome must contain an alliance of buyers n and $n - 1$.

Next, suppose that buyer n prefers the alliance $(n; n - 1; n - 2)$ to any other outcome. Because buyers n and $n - 1$ receive equal shares of savings and have equal preferences for alliances of all sizes, this implies that buyer $n - 1$ also prefers the alliance $(n; n - 1; n - 2)$ to any other outcome. Preferences of buyer $n - 2$ coincide with preferences of buyers n and $n - 1$ in all cases except when two-buyer alliances are considered, when he prefers alliance $(n; n - 2)$. Because savings in this alliance are smaller than savings in alliance $(n; n - 1)$, and buyer n prefers $(n; n - 1; n - 2)$ to $(n; n - 1)$, we can conclude that buyer $n - 2$ prefers $(n; n - 1; n - 2)$ to any other outcome. Therefore, every stable outcome must contain an alliance of buyers $n, n - 1$, and $n - 2$. A similar analysis can be performed for remaining preferences of retailer n . ■

Proof of Proposition 2: Buyer n (weekly) prefers the alliance $(n; n - 1; n - 2)$ over the alliance $(n; n - 1)$ if and only if

$$\frac{\sum_{i=n-2}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=n-2}^n q_j) \sum_{i=n-2}^n q_j}{3} \geq \frac{\sum_{i=n-1}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=n-1}^n q_j) \sum_{i=n-1}^n q_j}{2},$$

which can be simplified to

$$q_{n-2} \geq \frac{q_{n-1}q_n}{2(q_{n-1} + q_n)}.$$

Given that buyer n to prefer the alliance $(n; n-1; n-2)$ over the alliance $(n; n-1)$, he prefers $(n; n-1; n-2; n-3)$ over $(n; n-1; n-2)$ if and only if

$$\frac{\sum_{i=n-3}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=n-3}^n q_j) \sum_{i=n-3}^n q_j}{4} \geq \frac{\sum_{i=n-2}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=n-2}^n q_j) \sum_{i=n-2}^n q_j}{3},$$

which can be simplified to

$$q_{n-3} \geq \frac{q_{n-1}q_n + q_{n-1}q_{n-2} + q_{n-2}q_n}{3(q_{n-2} + q_{n-1} + q_n)}.$$

In a similar way we can verify that buyer n prefers the grand coalition to any other outcome if and only if for each $1 \leq k \leq n-2$,

$$q_k \geq \frac{\sum_{i>j} \sum_{j>k} q_i q_j}{(n-k) \sum_{j>k} q_j}.$$

Now, we find simple sufficient conditions for the above inequality. Because $(q_{n-1} + q_n)^2 \geq 4q_{n-1}q_n$, we have $\frac{q_{n-1}+q_n}{8} \geq \frac{q_{n-1}q_n}{2(q_{n-1}+q_n)}$. Because $q_{n-2}^2 + q_{n-1}^2 + q_n^2 \geq q_{n-1}q_n + q_{n-1}q_{n-2} + q_{n-2}q_n$ (by the Cauchy-Schwarz inequality), we have $(q_{n-2} + q_{n-1} + q_n)^2 \geq 3(q_{n-1}q_n + q_{n-1}q_{n-2} + q_{n-2}q_n)$ and thus $\frac{q_{n-2}+q_{n-1}+q_n}{9} \geq \frac{q_{n-1}q_n+q_{n-1}q_{n-2}+q_{n-2}q_n}{3(q_{n-2}+q_{n-1}+q_n)}$. Because $(\sum_{j>k} q_j)^2 \geq 2 \sum_{i>j} \sum_{j>k} q_i q_j$, we have $\frac{\sum_{j>k} q_j}{2(n-k)} \geq \frac{\sum_{i>j} \sum_{j>k} q_i q_j}{(n-k) \sum_{j>k} q_j}$. These lead to the sufficient condition in the proposition. \blacksquare

Proof of Proposition 3: Let us first consider two-buyer alliances, (ij) . Given buyer i with quantity q_i , the first derivative of the proportional allocation to buyer i with respect to q_j , the quantity contributed by buyer j , is given by

$$\begin{aligned} \frac{\partial \varphi^P(q_i, q_j)}{\partial q_j} &= \frac{\partial}{\partial q_j} \left[q_i \frac{p(q_i)q_i + p(q_j)q_j - p(q_i + q_j)(q_i + q_j)}{q_i + q_j} \right] \\ &= q_i \frac{[p'(q_j)q_j + p(q_j) - p'(q_i + q_j)(q_i + q_j)](q_i + q_j) - [p(q_i)q_i + p(q_j)q_j]}{(q_i + q_j)^2} \\ &= q_i \frac{[p'(q_j)q_j - p'(q_i + q_j)(q_i + q_j)](q_i + q_j) + [p(q_j) - p(q_i)]q_i}{(q_i + q_j)^2}. \end{aligned}$$

1. If $p'(q)q$ is increasing, the first term in the numerator is negative. Whenever $q_j > q_i$, $p(q_j) < p(q_i)$, and the second term in the numerator is also negative. Thus, whenever $q_j > q_i$, proportional allocations to buyer i are decreasing in buyer j 's contribution. Because of that, buyer 1 prefers an alliance with buyer 2 to any other two-buyer alliance. Similarly, buyer i prefers an alliance with a buyer who contributes less than i whenever $i > 2$. Consequently, both buyers 1 and 2 prefer (12) to any other two-retailer alliance.

Next, consider three-buyer alliances. Similar analysis to the one performed with two-buyer alliances shows that both buyers 1 and 2 prefer (123) to any other three-retailer alliance, while

buyer i , $i > 2$ prefers alliance $(i-2; i-1; i)$. Analogous conclusions can be made for any alliance size. Thus, buyer 1 prefers one of the following alliances: $(12), (123), \dots, (123 \dots n-1), N$.

Suppose that buyer 1 prefers the alliance (12) to any other outcome. This outcome is likely to happen when contributions of buyers 1 and 2 are not too far apart, while contributions of “stronger” buyers exceed significantly contributions of 1 and 2. As a result, allocation of buyer 1 in remaining alliances decreases. In such a case, buyer 2 also prefers alliance (12) to any other outcome. As their savings do not depend on actions of other buyers, the other buyers may not induce them to defect from this alliance, hence every stable outcome must contain an alliance of buyers 1 and 2.

Next, suppose that buyer 1 prefers the alliance (123) to any other outcome. This is likely to happen when contributions of buyers 1, 2, and 3 are close, while contributions of “stronger” buyers exceed significantly their contribution. In such a case, buyers 2 and 3 also prefer alliance (123) to any other outcome, hence every stable outcome must contain (123) . A similar analysis can be performed for remaining preferences of retailer 1.

2. If the quantity discount scheme is given by $p(q) = \alpha - \beta q^\eta$, $0 < \eta < 1$, then $p'(q)q = -\beta\eta q^\eta$, and

$$\begin{aligned} \frac{\partial \varphi^P(q_i, q_j)}{\partial q_j} &= q_i \frac{[-\beta\eta q_j^\eta + \beta\eta(q_i + q_j)^\eta](q_i + q_j) + [-\beta q_j^\eta + \beta q_i^\eta] q_i}{(q_i + q_j)^2} \\ &= \beta q_i \frac{-\eta q_j^\eta (q_i + q_j) + \eta (q_i + q_j)^{\eta+1} - q_i q_j^\eta + q_i^{\eta+1}}{(q_i + q_j)^2} \\ &= \frac{\beta q_i^{\eta+2}}{(q_i + q_j)^2} \left(-\eta \left(1 + \frac{q_j}{q_i}\right) \left(\frac{q_j}{q_i}\right)^\eta + \eta \left(1 + \frac{q_j}{q_i}\right)^{\eta+1} - \left(\frac{q_j}{q_i}\right)^\eta + 1 \right). \end{aligned}$$

Thus, $\frac{\partial \varphi^P(q_i, q_j)}{\partial q_j} \geq 0$ if and only if $\eta \left(1 + \frac{q_j}{q_i}\right) \left(\left(1 + \frac{q_j}{q_i}\right)^\eta - \left(\frac{q_j}{q_i}\right)^\eta\right) - \left(\frac{q_j}{q_i}\right)^\eta + 1 \geq 0$. It can be verified that the inequality holds if and only if $\frac{q_j}{q_i} \leq \lambda(\eta)$ where $\lambda(\eta)$ is the root of $\eta(1 + \lambda)((1 + \lambda)^\eta - \lambda^\eta) - \lambda^\eta + 1$. Clearly, $\frac{q_n}{q_1} \leq \lambda(\eta)$ implies $\frac{q_j}{q_i} \leq \lambda(\eta)$ for all i and j . The rest of the proof can be completed by an analysis similar to that used in the proof of Proposition 1.

3. If the discount scheme is linear, then

$$\frac{\partial \varphi^P(q_i, q_j)}{\partial q_j} = q_i \frac{[-\bar{\beta} q_j + \bar{\beta}(q_i + q_j)](q_i + q_j) + [\bar{\alpha} - \bar{\beta} q_j - \bar{\alpha} + \bar{\beta} q_i] q_i}{(q_i + q_j)^2} = \frac{2\bar{\beta} q_i^3}{(q_i + q_j)^2} > 0,$$

hence proportional allocations to buyer i are increasing in buyer j 's contribution. Thus, buyers prefer alliances with players who contribute more to the quantity purchased.

Buyer n prefers the alliance $(n; \dots; k+1; k)$ over the alliance $(n; \dots; k+1)$ for any $k \leq n-2$ if and only if

$$\begin{aligned}
& q_n \frac{\sum_{i=k}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=k}^n q_i) \sum_{i=k}^n q_i}{\sum_{i=k}^n q_i} \\
& \geq q_n \frac{\sum_{i=k+1}^n [q_i(\bar{\alpha} - \bar{\beta}q_i)] - (\bar{\alpha} - \bar{\beta} \sum_{i=k+1}^n q_i) \sum_{i=k+1}^n q_i}{\sum_{i=k+1}^n q_i}, \\
\text{i.e., } & \left(\left(\sum_{i=k}^n q_i \right)^2 - \sum_{i=k}^n q_i^2 \right) \sum_{i=k+1}^n q_i \geq \left(\left(\sum_{i=k+1}^n q_i \right)^2 - \sum_{i=k+1}^n q_i^2 \right) \sum_{i=k}^n q_i, \\
& \left(\sum_{i=k}^{n-1} \sum_{j=i+1}^n q_i q_j \right) \sum_{i=k+1}^n q_i \geq \left(\sum_{i=k+1}^{n-1} \sum_{j=i+1}^n q_i q_j \right) \sum_{i=k}^n q_i, \\
& \left(\sum_{j=k+1}^n q_k q_j \right) \sum_{i=k+1}^n q_i \geq \left(\sum_{i=k+1}^{n-1} \sum_{j=i+1}^n q_i q_j \right) q_k, \\
& \left(\sum_{i=k+1}^n q_i \right)^2 \geq \sum_{i=k+1}^{n-1} \sum_{j=i+1}^n q_i q_j.
\end{aligned}$$

The last inequality is true for all $k \leq n-2$. Thus, buyer n prefers the grand coalition to any other outcome, and we can use an analysis similar to that used in the proof of Proposition 1 to complete the proof. \blacksquare

Proof of Theorem 1: The profit game is convex if $\hat{v}(T \cup \{i\}) - \hat{v}(T) \geq \hat{v}(S \cup \{i\}) - \hat{v}(S)$ whenever $S \subset T \subseteq N \setminus \{i\}$.

Consider an arbitrary coalition $V \subseteq N \setminus \{i\}$:

$$\begin{aligned}
\hat{v}(V \cup \{i\}) - \hat{v}(V) &= (r - \hat{p}^{V \cup \{i\}}) \hat{Q}^{V \cup \{i\}} - \sum_{j \in V \cup \{i\}} (r - \hat{p}_j) \hat{q}_j - (r - \hat{p}^V) \hat{Q}^V + \sum_{j \in V} (r - \hat{p}_j) \hat{q}_j \\
&= (r - \hat{p}^{V \cup \{i\}}) \hat{Q}^{V \cup \{i\}} - (r - \hat{p}^V) \hat{Q}^V - (r - \hat{p}_i) \hat{q}_i.
\end{aligned}$$

Thus, the above equation implies that the profit game is convex if and only if

$$\begin{aligned}
& (r - \hat{p}^{T \cup \{i\}}) \hat{Q}^{T \cup \{i\}} - (r - \hat{p}^T) \hat{Q}^T \geq (r - \hat{p}^{S \cup \{i\}}) \hat{Q}^{S \cup \{i\}} - (r - \hat{p}^S) \hat{Q}^S \iff \\
& (\hat{p}^T - \hat{p}^{T \cup \{i\}}) \hat{Q}^T + (r - \hat{p}^{T \cup \{i\}}) (\hat{Q}^{T \cup \{i\}} - \hat{Q}^T) \geq (\hat{p}^S - \hat{p}^{S \cup \{i\}}) \hat{Q}^S + (r - \hat{p}^{S \cup \{i\}}) (\hat{Q}^{S \cup \{i\}} - \hat{Q}^S)
\end{aligned} \tag{14}$$

Consider the decrease in price observed when buyer i joins coalition V :

$$\begin{aligned}
\hat{p}^V - \hat{p}^{V \cup \{i\}} &= \frac{\bar{\alpha} - \bar{\beta} \sum_{j \in V} a_j}{1 - \bar{\beta} \sum_{j \in V} b_j} - \frac{\bar{\alpha} - \bar{\beta} \sum_{j \in V} a_j - \bar{\beta} a_i}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} \\
&= (\bar{\alpha} - \bar{\beta} \sum_{j \in V} a_j) \left[\frac{1}{1 - \bar{\beta} \sum_{j \in V} b_j} - \frac{1}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} \right] + \frac{\bar{\beta} a_i}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} \\
&= \frac{\bar{\beta} a_i}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} - (\bar{\alpha} - \bar{\beta} \sum_{j \in V} a_j) \frac{\bar{\beta} b_i}{(1 - \bar{\beta} \sum_{j \in V} b_j)(1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i)} \\
&= \frac{\bar{\beta}}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} \left[a_i - b_i \frac{\bar{\alpha} - \bar{\beta} \sum_{j \in V} a_j}{1 - \bar{\beta} \sum_{j \in V} b_j} \right] \\
&= \frac{\bar{\beta}}{1 - \bar{\beta} \sum_{j \in V} b_j - \bar{\beta} b_i} [a_i - b_i \hat{p}^V].
\end{aligned}$$

It follows from above that $\hat{p}^T - \hat{p}^{T \cup \{i\}} > \hat{p}^S - \hat{p}^{S \cup \{i\}}$ for $S \subset T$. Moreover, because

$$\hat{Q}^{V \cup \{i\}} - \hat{Q}^V = \frac{\hat{p}^V - \hat{p}^{V \cup \{i\}}}{\bar{\beta}},$$

we have $\hat{Q}^{T \cup \{i\}} - \hat{Q}^T > \hat{Q}^{S \cup \{i\}} - \hat{Q}^S$. Together with Proposition 5, this implies that (14) holds, and the game is convex. \blacksquare

Proof of Proposition 7: Let us first consider two-buyer alliances, $S = \{i, j\}$. Suppose that the function $D_j(p) = a_j - b_j p$ changes to $D'_j(p) = a'_j - b'_j p$, $a'_j \geq a_j$, $b'_j \geq b_j$, so that $D'_j(\hat{p}_j) = \hat{q}_j + \Delta$ for some $\Delta > 0$. In other words, at the price that was optimal before the change, retailer j increases his ordering quantity (see Figure A1).

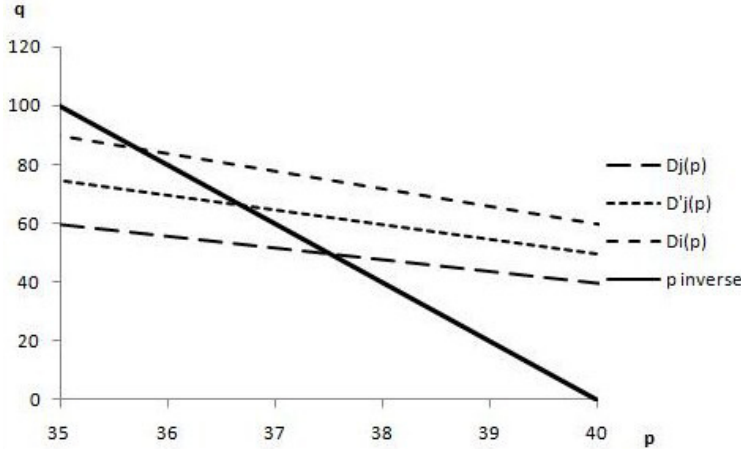


Figure A1: Optimal ordering quantities for different demand functions.

It is straightforward to calculate new optimal price and quantity for buyer j if he acts alone:

$$\hat{p}'_j = \hat{p}_j - \frac{\bar{\beta} \Delta}{1 - \bar{\beta} b'_j} \leq \hat{p}_j, \quad \hat{q}'_j = \hat{q}_j + \frac{\Delta}{1 - \bar{\beta} b'_j} \geq \hat{q}_j,$$

and the optimal price and quantity for the coalition of buyers i and j , $S = \{i, j\}$:

$$\begin{aligned}\hat{p}'^S &= \hat{p}^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} - \frac{\bar{\beta} \left[(b'_j - b_j) \hat{p}_j + \Delta \right]}{1 - \bar{\beta}(b_i + b'_j)}, \\ \hat{Q}'^S &= \hat{Q}^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)}, \\ \hat{Q}'^S_i &= \hat{Q}'^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)}, \\ \hat{Q}'^S_j &= \hat{Q}'^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)}.\end{aligned}$$

The new allocation to buyer i can be expressed as

$$\begin{aligned}\varphi_i'^S &= \frac{\hat{Q}'^S_i \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)}}{\hat{Q}'^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)}} \\ &= \left[\left(r - \hat{p}^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\bar{\beta} \left[(b'_j - b_j) \hat{p}_j + \Delta \right]}{1 - \bar{\beta}(b_i + b'_j)} \right) \left(\hat{Q}'^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)} \right) - \right. \\ &\quad \left. (r - \hat{p}_i) \hat{q}_i - \left(r - \hat{p}_j + \frac{\bar{\beta} \Delta}{1 - \bar{\beta} b'_j} \right) \left(\hat{q}_j + \frac{\Delta}{1 - \bar{\beta} b'_j} \right) \right].\end{aligned}$$

Note that

$$\begin{aligned}\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j) &= \Delta - \bar{\beta} \hat{q}_j (b'_j - b_j) + \bar{\alpha} b'_j - \bar{\alpha} b_j \\ &= \Delta + b'_j \hat{p}_j + \bar{\beta} \hat{q}_j b_j - \bar{\alpha} b'_j + \bar{\alpha} b_j - \bar{\alpha} b_j \\ &= \Delta + b'_j \hat{p}_j - b_j \hat{p}_j - \bar{\alpha} (b'_j - b_j) \\ &= \Delta + (\hat{p}_j - \bar{\alpha}) (b'_j - b_j) \\ &= (\hat{q}'_j - \hat{q}_j) (1 - \bar{\beta} b'_j) - \bar{\beta} \hat{q}_j (b'_j - b_j) \\ &= \hat{q}'_j (1 - \bar{\beta} b'_j) - \hat{q}_j (1 - \bar{\beta} b_j) \\ &\leq \hat{q}'_j \bar{\beta} (b_j - b'_j) \leq 0,\end{aligned}$$

because $\hat{q}'_j \geq \hat{q}_j$ and $b'_j \geq b_j$. This implies that

$$\frac{\hat{Q}'^S_i \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)}}{\hat{Q}'^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)}} = \frac{\hat{Q}'^S_i (1 - \bar{\beta}(b_i + b_j))}{\hat{Q}'^S (1 - \bar{\beta}(b_i + b_j)) + (\Delta - \bar{\beta} \hat{q}_j (b'_j - b_j))} \geq \frac{\hat{Q}'^S_i}{\hat{Q}'^S}.$$

We now evaluate the relationship between the new and the old allocation to buyer i :

$$\begin{aligned}
\frac{\hat{Q}_i^S}{\hat{Q}_i^S} (\varphi'_i - \varphi_i) &\geq \left(r - \hat{p}^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\bar{\beta} [(b'_j - b_j)\hat{p}_j + \Delta]}{1 - \bar{\beta}(b_i + b'_j)} \right) \left(\hat{Q}^S \frac{1 - \bar{\beta}(b_i + b_j)}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\Delta - \bar{\beta}\hat{q}_j(b'_j - b_j)}{1 - \bar{\beta}(b_i + b'_j)} \right) - \\
&\quad (r - \hat{p}_i)\hat{q}_i - \left(r - \hat{p}_j + \frac{\bar{\beta}\Delta}{1 - \bar{\beta}b'_j} \right) \left(\hat{q}_j + \frac{\Delta}{1 - \bar{\beta}b'_j} \right) - \\
&\quad (r - \hat{p}^S)\hat{Q}^S + (r - \hat{p}_i)\hat{q}_i + (r - \hat{p}_j)q_j \\
&= \left(r - \hat{p}^S + \frac{(b'_j - b_j)(\hat{p}_j - \hat{p}^S) + \bar{\beta}\Delta}{1 - \bar{\beta}(b_i + b'_j)} \right) \left(\hat{Q}^S + \frac{(b'_j - b_j)\bar{\beta}(\hat{Q}^S - \hat{q}_j) + \Delta}{1 - \bar{\beta}(b_i + b'_j)} \right) - \\
&\quad \frac{\bar{\beta}\Delta}{1 - \bar{\beta}b'_j} \left(\hat{q}_j + \frac{\Delta}{1 - \bar{\beta}b'_j} \right) - (r - \hat{p}_j) \frac{\Delta}{1 - \bar{\beta}b'_j} - (r - \hat{p}^S)\hat{Q}^S \\
&\geq (r - \hat{p}^S) \frac{\Delta}{1 - \bar{\beta}(b_i + b'_j)} + \frac{\bar{\beta}\Delta^2}{(1 - \bar{\beta}(b_i + b'_j))^2} + \frac{\bar{\beta}\Delta}{1 - \bar{\beta}(b_i + b'_j)}\hat{Q}^S - \\
&\quad \frac{\bar{\beta}\Delta}{1 - \bar{\beta}b'_j}\hat{q}_j - \frac{\bar{\beta}\Delta^2}{(1 - \bar{\beta}b'_j)^2} - (r - \hat{p}_j) \frac{\Delta}{1 - \bar{\beta}b'_j} \geq 0,
\end{aligned}$$

because $Q^S \geq \hat{q}_j$ and $\hat{p}^S \leq \hat{p}_j$. Thus, when buyer i joins a buyer who orders a larger quantity when acting alone, the allocation to buyer i increases. Consequently, we have situation similar to the one described in the model in which each buyer receives equal share of benefits—buyers prefer alliances with players who contribute more to the quantity purchased. We can, therefore, use an analysis similar to that used in the proof of Proposition 1 to complete the proof. \blacksquare

Proof of Proposition 9:

1. The proof under equal allocations is straightforward, and therefore omitted.

Note that the proportional allocation allocates to buyer i in coalition $S = \{i, j\}$ amount

$$\begin{aligned}
\varphi_i &= \frac{\hat{Q}_i^S}{\hat{Q}^S} \left[(\hat{r}^S - \hat{p}^S)\hat{Q}^S - (\hat{r}_i - \hat{p}_i)\hat{q}_i - (\hat{r}_j - \hat{p}_j)\hat{q}_j \right] \\
&= \frac{a_i - \bar{\alpha}b_i}{4(a_i + a_j - \bar{\alpha}(b_i + b_j))} \left[\frac{(a_i + a_j - \bar{\alpha}(b_i + b_j))^2}{(b_i + b_j)(1 - \bar{\beta}(b_i + b_j))} - \frac{(a_i - \bar{\alpha}b_i)^2}{b_i(1 - \bar{\beta}b_i)} - \frac{(a_j - \bar{\alpha}b_j)^2}{b_j(1 - \bar{\beta}b_j)} \right].
\end{aligned}$$

If we assume that $\frac{a_i}{b_i} = C$, then $a_i - \bar{\alpha}b_i = b_i(C - \bar{\alpha})$, and the above allocation can be rewritten as

$$\varphi_i = \frac{b_i(C - \bar{\alpha})^2}{4(b_i + b_j)} \left[\frac{b_i + b_j}{1 - \bar{\beta}(b_i + b_j)} - \frac{b_i}{1 - \bar{\beta}b_i} - \frac{b_j}{1 - \bar{\beta}b_j} \right].$$

Now, suppose that buyer j increases his order, which corresponds to some $b'_j > b_j$; then, the new allocation to buyer i becomes

$$\varphi'_i = \frac{b_i(C - \bar{\alpha})^2}{4(b_i + b'_j)} \left[\frac{b_i + b'_j}{1 - \bar{\beta}(b_i + b'_j)} - \frac{b_i}{1 - \bar{\beta}b_i} - \frac{b'_j}{1 - \bar{\beta}b'_j} \right].$$

Thus, we have

$$\begin{aligned}
\frac{4(\varphi'_i - \varphi_i)}{b_i(C - \bar{\alpha})^2} &= \frac{1}{1 - \bar{\beta}(b_i + b'_j)} - \frac{b_i}{(b_i + b'_j)(1 - \bar{\beta}b_i)} - \frac{b'_j}{(b_i + b'_j)(1 - \bar{\beta}b'_j)} \\
&\quad - \frac{1}{1 - \bar{\beta}(b_i + b_j)} + \frac{b_i}{(b_i + b_j)(1 - \bar{\beta}b_i)} - \frac{b_j}{(b_i + b_j)(1 - \bar{\beta}b_j)} \\
&= \frac{\bar{\beta}(b'_j - b_j)}{[1 - \bar{\beta}(b_i + b_j)][1 - \bar{\beta}(b_i + b'_j)]} + \frac{b_i(b'_j - b_j)}{(1 - \bar{\beta}b_i)(b_i + b_j)(b_i + b'_j)} \\
&\quad - \frac{(b_i + \bar{\beta}b_j b'_j)(b'_j - b_j)}{(1 - \bar{\beta}b_j)(1 - \bar{\beta}b'_j)(b_i + b_j)(b_i + b'_j)}.
\end{aligned}$$

After some tedious calculation, we can obtain

$$\frac{4(\varphi'_i - \varphi_i)}{\bar{\beta}b_i^2(C - \bar{\alpha})^2(b'_j - b_j)} = \frac{g(\bar{\beta}, b_i, b_j, b'_j)}{[1 - \bar{\beta}(b_i + b_j)][1 - \bar{\beta}(b_i + b'_j)](1 - \bar{\beta}b_i)(1 - \bar{\beta}b_j)(1 - \bar{\beta}b'_j)(b_i + b_j)(b_i + b'_j)},$$

where

$$\begin{aligned}
g(\bar{\beta}, b_i, b_j, b'_j) &= 2b_i + \bar{\beta}[3b_j b'_j - b_i^2 - b_i(b_i + b_j) - b_i(b_i + b'_j)] \\
&\quad + \bar{\beta}^2[b_i(b_i + b_j)(b_i + b'_j) - 2b_j b'_j(b_i + b_j) - 2b_j b'_j(b_i + b'_j)] + \bar{\beta}^3 b_j b'_j(b_i + b_j)(b_i + b'_j).
\end{aligned}$$

In order to show that allocation of buyer i increases in the size of buyer j 's allocation, we need to show that $g(\bar{\beta}, b_i, b_j, b'_j) \geq 0$. Recall that $1 - \bar{\beta}(b_i + b'_j) \geq 0$; then, we can see that

$$\begin{aligned}
g(\bar{\beta}, b_i, b_j, b'_j) &\geq b_i + 3\bar{\beta}b_j b'_j - \bar{\beta}b_i^2 - \bar{\beta}b_i(b_i + b_j) \\
&\quad + \bar{\beta}^2 b_i(b_i + b_j)(b_i + b'_j) - 2\bar{\beta}^2 b_j b'_j(b_i + b_j) - 2\bar{\beta}b_j b'_j + \bar{\beta}^3 b_j b'_j(b_i + b_j)(b_i + b'_j) \\
&= b_i[1 - \bar{\beta}b_i - \bar{\beta}(b_i + b_j) + \bar{\beta}^2(b_i + b_j)(b_i + b'_j)] \\
&\quad + \bar{\beta}b_j b'_j[1 - 2\bar{\beta}(b_i + b_j) + \bar{\beta}^2(b_i + b_j)(b_i + b'_j)] \\
&\geq (b_i + \bar{\beta}b_j b'_j)[1 - \bar{\beta}(b_i + b_j)]^2 \geq 0.
\end{aligned}$$

We can now use an analysis similar to that used in the proof of Proposition 1 to complete the proof of this item.

2. It is easy to show that the game is convex because the margin is equal for any buyer or any GPO, and the quantity purchased increases after a GPO is formed. ■